A Rapid Approach for Calculating the Damped Eigenvalues of a Gas Turbine on a Minicomputer: Theory

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The calculation of the damped eigenvalues of a large multistation gas turbine by the complex matrix transfer procedure may encounter numerical difficulties, even on a large computer due to numerical round-off errors. In this paper, a procedure is presented in which the damped eigenvalues may be rapidly and accurately calculated on a minicomputer with accuracy which rivals that of a mainframe computer using the matrix transfer method. The method presented in this paper is based upon the use of constrained normal modes plus the rigid body modes in order to generate the characteristic polynomial of the system. The constrained undamped modes, using the matrix transfer process with scaling, may be very accurately calculated for a multistation turbine on a minicomputer. In this paper, a five station rotor is evaluated to demonstrate the procedure. A method is presented in which the characteristic polynomial may be automatically generated by Leverrier's algorithm. The characteristic polynomial may be directly solved or the coefficients of the polynomial may be examined by the Routh criteria to determine stability. The method is accurate and easy to implement on a 16 bit minicomputer.

I Background and Introduction

In the analysis of the dynamic characteristics of high-speed rotating machinery, such as compressors and gas turbines, it is desirable to determine the damped eigenvalues of the system. The magnitude of the real coefficient of the damped eigenvalue determines the rotor amplification factor of the system. For example, if the rotor amplification factor is 10 or greater, the rotor system will be susceptible to low levels of unbalance excitation. The API code, for example, requires turbine and compressor amplification factors to be 8 or less. A more serious problem with rotating machinery at high speeds is the occurrence of self-excited whirl motion. Self-excited whirl motion or rotor instability may be caused by such factors as aerodynamic cross-coupling effects of the impellers, labyrinth and fluid film seals, and hydrodynamic journal bearings.

In the 1950's, the extent of rotor-bearing analysis consisted mainly of undamped critical speed determination. The major paper in this field was presented by M. Prohl. The calculation procedure that was required at the time was with a team of people working with a desk calculator involving days or weeks of work. Now, 100 station rotors or more may be rapidly analyzed on the minicomputer in a matter of minutes. A major contribution to the field of rotor-bearing stability was presented by Lund in 1974, when he described the complex matrix transfer procedure to calculate the stability characteristics of multistation turborotors, including generalized linear bearing coefficients. This paper represented a major advancement in the field of stability analysis. There are, however, inherent numerical difficulties associated with the matrix transfer method. In the absence of scaling of the transfer matrices, numerical round-off errors occur on large station systems which generate invalid eigenvalues. This procedure may be somewhat alleviated by using double precision and scaling of the matrices. However, it cannot be completely avoided. A paper on the scaling and rotor modeling process will be presented at a later time.

In this paper, a method based on constrained normal modes plus rigid body modes is presented to determine the damped eigenvalues of the system. The area of modal analysis is well developed and is extensively employed by structural engineers to simplify the dynamical representation of the system. One of the standard methods of modal analysis is to eliminate the damping or dissipation terms in the equations of motion and solve for the undamped normal modes of the system. By expressing the deflection as a sum of the undamped normal modes, the modal dynamical equations of motion may be generated. One of the typical assumptions in structural dynamics is that the modal damping cross-coupling terms are small and are thus eliminated. In the case of a gas turbine with hydrodynamic fluid film bearings or squeeze film dampers, the modal cross-coupling damping terms can never be eliminated. The assumption that the normal modal equations

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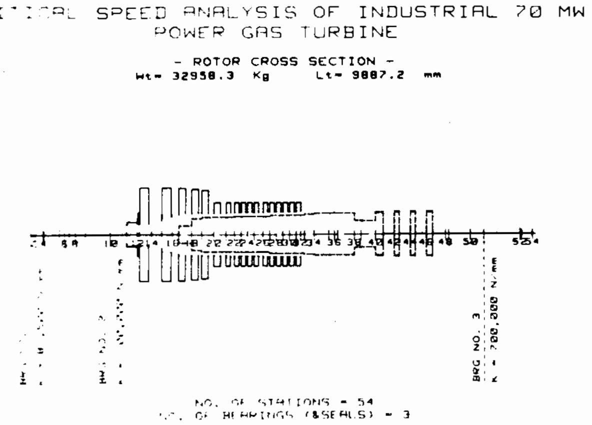

Fig. 1 Cross section of 54 station gas turbine

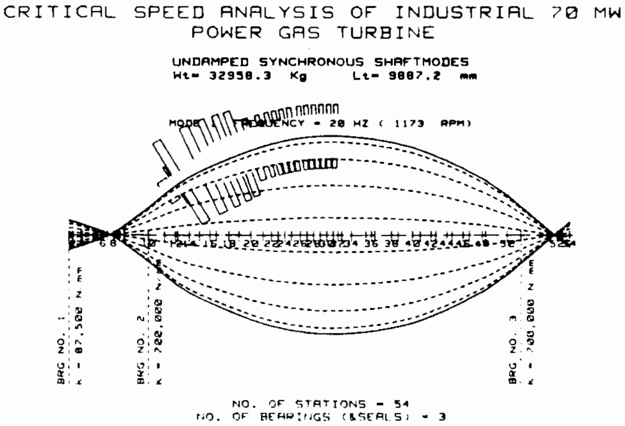

Fig. 2 Undamped first normal mode of gas turbine

of motion are uncoupled is based upon the approximation that the damping matrix is proportional to the mass or stiffness matrix. In the case of rotating machinery with bearings or seals, this situation never occurs in practice. It is only valid if the damping of the system is extremely light and of the order of only one or two percent of critical damping.

Figure 1, for example, represents a 72,000 lb (32,727 kg) gas turbine with 54 stations. The first two normal modes for this system are shown in Figs. 2 and 3. The normal modes, for example, were generated on a HP-9845B Desk Computer with a 16 bit processor. In an attempt to analyze the damped eigenvalues of this system on a mainframe computer, numerical difficulties were encountered with the matrix transfer procedure. The analysis of the 70 MW power generation gas turbine as shown in Fig. 1 will be presented in detail in Part II—Applications.

The method of modal analysis appears to be a very attractive procedure to describe the dynamical behavior of such a complex system. The area of modal analysis has received extensive treatment and classical descriptions of this method are given by the various researchers in structural dynamics, such as Hurty and Rubinstein. Modal analysis has been extensively applied to rotating machinery by Bishop, Parkinson, and Black in England and Childs, Nelson, Gunter, Choy, and Li, etc. in recent papers in the U.S. This is just to mention a few of the many papers in this area.

The procedure is attractive from the standpoint that the various system modes normally need to be calculated only once. Modes are then used as building blocks to describe the

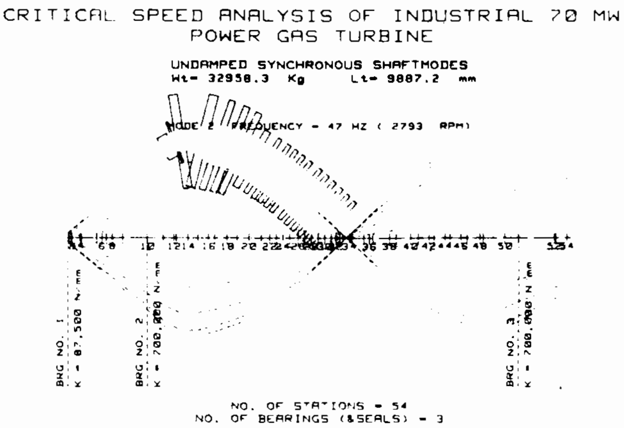

Fig. 3 Undamped second normal mode of gas turbine

generalized equations of motion usually in terms of undamped modes of motion.

There are various sets of mode shapes that may be employed as functional sets to span the vector space of equations. These mode sets may be roughly classified in the categories of normal, constrained, and free-free rigid body mode sets. Dynamic analysis may also be performed by the use of complex modes as outlined by Foss and also expanded upon by Lund. The undamped modes may be used to generate the complex damped modes; although a number of papers

Nomenclature

A_c = amplification factor = $1/2\xi$ (dim)
$[A]$ = $2n \times 2n$ mass and damping matrix
$[B]$ = $2n \times 2n$ mass and stiffness matrix
$[B]_k$ = kth Leverrier matrix
b = subscript, bearing
c = subscript, constraint mode
$[C]$ = damping, N-m/s
C_{ij} = modal cross-coupling coefficient
$\bar{C}$ = damping coefficient = $2C/M\omega_c$ (dim)
$[D]$ = $2n \times 2n$ dynamic system matrix (dim)
f = frequency ratio = $(\omega_r/\omega_c)^2$ = $2K_b/K_s$ (dim)
K_b = bearing stiffness, N/m
K_s = shaft stiffness, N/m
K = $2K_b/K_s$ = stiffness ratio (dim)
$[K]$ = $n \times n$ stiffness matrix, N/m
$[M]$ = $n \times n$ mass matrix, kg
M_i = modal mass, ith mode, kg
$\bar{M}_{ro}$ = normalized modal mass cross-coupling coefficient (dim)
n = order of system
p = real part of complex root, rad/s
q = generalized coordinate
q_c = generalized constraint coordinate
q_r = generalized rigid body coordinate
r = subscript, rigid body
s = complex root = $p + i\nu$, rad/s
$\{X\}$ = displacement vector, m
λ = complex inverse root, rad/s
ω = natural frequency, rad/s
ω_c = constrained natural frequency, rad/s
ω_r = rigid body natural frequency, rad/s
Ω = normalization factor, rad/s
ν = imaginary part of complex root, rad/s
$\{\phi\}$ = ith normal mode
Λ = normalized frequency = λ/ω_c (dim)
$\{\Phi\}$ = orthonormal mode

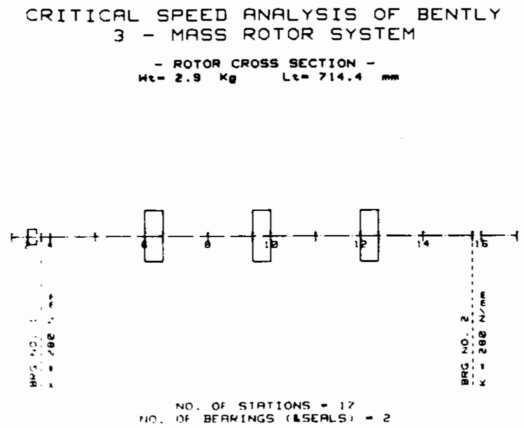

Fig. 4 Cross section of 17 station three-mass test rotor

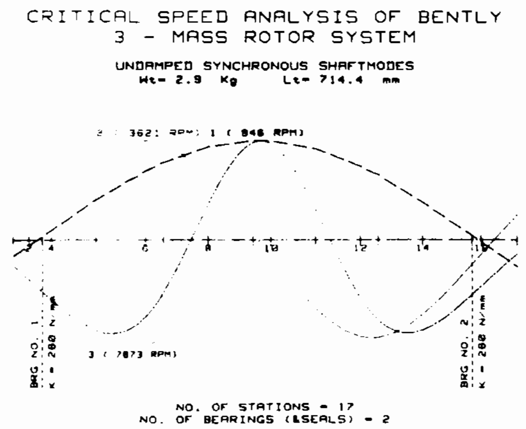

Fig. 5 Mode shapes of three-mass test rotor

have been written in various fields on modal analysis, few papers have been written on the errors generated by the truncation to a finite number of modes. Li and Gunter in 1981 presented one study on modal truncation error in component mode analysis of a dual rotor system.

However, considerable research is still required in this area. In this paper it is shown that the use of normal modes may not generate accurate results even if the modal errors coupling damping terms are retained. The use of component modes along with rigid body modes is shown to generate the exact eigenvalues. An excellent presentation on component mode analysis is given by Nelson.

The generation of the damped eigenvalue problem requires the initial solution of the undamped planar problem, assuming the bearings are node points. Once the rotor modal mass and frequencies are obtained, then the rotor elastic properties do not have to be further calculated. The entire stability analysis may be performed on a 16 bit minicomputer with extremely good accuracy.

II Damped Eigenvalue Analysis

1 General Equations of Motion. The introduction of linear viscous damping into a dynamical system results in the following general matrix formulation:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = F(t) \tag{1.1}$$

The generalized eigenvalue problem for the damped system is formulated by setting $F(t) = 0$ and assuming the displacement vector $\{x\}$ to be of the form

$$\{x\} = \{X\}e^{\lambda t} \tag{1.2}$$

The generalized eigenvalue problem may be written as

Table 2.1 Three mass system mode shapes and eigenvalues

I		Undamped Displacement Modes		
FREQ	r/min (Hz) rad/s	968.6 (16.1) 101.42	3618 (60.3) 378.85	7911 (131.9) 828.4
M Modal	Lb-sec^2/in. (kg)	.0095 (1.666)	.0093 (1.632)	.0091 (1.597)
Station	1	.057	0.334	-0.608
	2	0.724	1.000	-0.6906
	3	1.000	0.0	1.00
	4	0.724	-1.000	-0.6906
	5	0.057	-0.334	-0.608

II		Constrained Normal Mode		
FREQ	r/min (Hz) rad/s	995 (16.6) 104.17	3947 (66) 413.3	8470 (141.2) 886.9
M Modal	Lb-sec^2/in. (kg)	.0093 (1.632)	.0093 (1.632)	.0093 (1.632)
Station	2	0.7071	1.00	-0.7071
	3	1.000	0.00	1.000
	4	0.7071	1.00	-0.7071

III		Free-Free Modes		
FREQ	r/min (Hz) rad/s	0	0	4870.5 (81.2) 510
M Modal	Lb-sec^2/in. (kg)	.01395 (2.448)	0.0093 (1.632)	.0069 (1.211)
Station	1	1	2	-3.00
	2	1	1	-0.50
	3	1	0	1.00
	4	1	-1	-0.50
	5	1	-2	3.00

$$[\lambda^2[M] + \lambda[C] + [K]]\{X\} = 0 \tag{1.3}$$

Since $\{X\}$ is in general a nonzero vector, Cramer's rule requires that the determinant of the coefficients must vanish. This leads to the following equation:

$$|\lambda^2[M] + \lambda[C] + [K]| = 0 \tag{1.4}$$

Equation (1.4) represents a polynomial of the form

$$P(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) - - (\lambda-\lambda_{2n}) \tag{1.5}$$

In general, the roots λ_i are complex for underdamped systems. Since the $[M]$, $[C]$, and $[K]$ matrices are all real coefficients, the coefficients of the characteristic equation are all real numbers. The complex roots λ_i have a corresponding complex conjugate root $\bar{\lambda}_i$.

For a full system of n degrees of freedom (no zeros in the mass matrix), the order of the polynomial is $2n$. In the case where none of the roots are critically damped, the characteristic polynomial is of the form:

$$P(\lambda) = (\lambda-\lambda_1)(\lambda-\bar{\lambda}_1)(\lambda-\lambda_2)(\lambda-\bar{\lambda}_2) - (\lambda-\lambda_n)(\lambda-\bar{\lambda}_n) \tag{1.6}$$

The root λ_i is of the general form:

$$\lambda_i = P_i + i\nu_i \text{ (rad/s)}$$
$$\bar{\lambda}_i = P_i - i\nu_i \tag{1.7}$$

The resulting motion corresponding to the ith root is of the form:

$$\{x\} = \{X\}_i e^{P_i t}[\cos\nu_i t + i\sin\nu_i t] \tag{1.8}$$

Hence if the real component P of the complex root λ is greater than zero, the system motion grows exponentially with time and the system is said to be unstable in the linear sense.

Example 1. Consider the three-mass system as shown in Fig. 4. Damping coefficients of $C = 1$ lb-s/in. (175 N-s/in.) are applied at each bearing. The 17 station-3 mass test rotor as shown in Fig. 4 may be represented by the five degree of freedom system as follows:

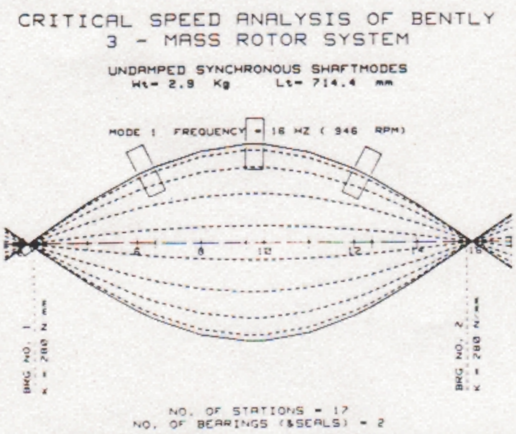

Fig. 6 Animated first mode of three-mass test rotor

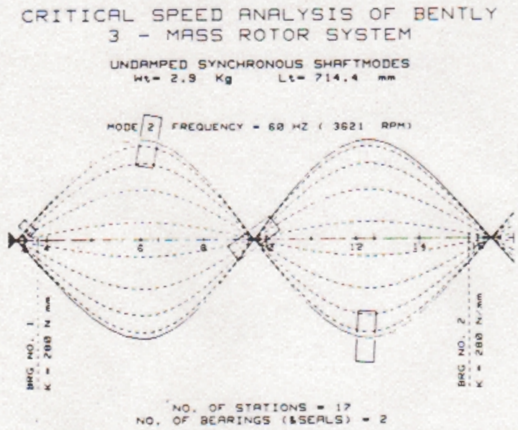

Fig. 7 Animated second mode of three-mass test rotor

$$M\lambda^2 \begin{bmatrix} 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \end{bmatrix} + C\lambda \begin{bmatrix} 1&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \end{bmatrix}$$

$$+K \begin{bmatrix} 1.215 & -0.489 & 0.348 & -0.089 & 0.015 \\ & 1.326 & -1.274 & 0.526 & -0.089 \\ & & 1.852 & -1.274 & 0.348 \\ & (\mathrm{SYM}) & & 1.326 & -0.489 \\ & & & & 1.215 \end{bmatrix} = 0 \quad (1.9)$$

where the coefficients M, C, and K are given by:

$M = 0.00465$ lb-s²/in. (0.815 kg)
$C = 1.00$ lb-s/in. (175 N-s/in.)
$K = 1{,}000$ lb/in. (175,000 N-s/in.)

In the above formulation, the $[M]$, $[C]$, and $[K]$ matrices are symmetric and positive definite (Hermitian) form. The eigenvalues of the system will all be of the form:

$$\lambda_i = -P_i + i\nu_i \quad (1.10)$$

There can be no real positive roots in the system as given in Example 1.

2 Formulation of the Damped Eigenvalue Problem by Normal Modal Analysis. In general, the formulation of the damped eigenvalue problem as given by equation (1.3) is difficult to numerically evaluate. The complexity of the system may be reduced by first using normal modes and then by the constrained modal method. It is possible to directly evaluate the complex eigenvalues from the general formulation. There are several methods that have been successfully employed in large structural dynamics analysis codes such as NASTRAN. One of these methods is the inverse power method with shifts and triangular decomposition.

In equation (1.4), the order of the system may be 10,000 degrees of freedom for a large structural problem. It is therefore obviously impractical to determine the characteristic equation for the complete system. The order of the system may be reduced to practical limits by employing the undamped natural frequencies and mode shapes of the system.

Table 2.1 represents three sets of modes obtained for Example 1 with no damping. The first set of modes represent the normal modes with a bearing support spring rate of 1,000 lb/in. (175,000 N/m) located at the bearings as shown in Fig. 4. Figure 5 represents the first three normal mode shapes for this three-mass model. Figures 6 and 7 represent the first and second animated mode shapes for the three-mass system. The second set of modes represent the constrained normal modes. These modes are obtained by constraining the motion at the support locations. The third set of mode functions represent the free-free modes. These modes are obtained by assuming no support restraint acts at the end of the station. The first two modes of this set are rigid body modes of zero frequency and the third mode is the system first free-free bending mode.

The modes listed in Table 2.1 are called displacement modes. The maximum value of the displacement mode is unity at a mass station. Since Guyan reduction was used to determine the free-free modes, the displacement at the bearing locations, which are massless, is greater than unity.

The system equations of motion are reduced by assuming the displacement vector $\{X\}$ may be represented in terms of the normal modes (I of Table 2.1) as follows:

$$\{X\} = \sum_{j=1}^{n} q_j \{\phi\}_j \quad (2.1)$$

where N = order of system,
q_j = jth generalized coordinate.

Substituting equation (2.1) into equation (1.1) results in:

$$[M]\sum_{j=1}^{n} \ddot{q}_j\{\phi\}_j + [C]\sum_{j=1}^{n} \dot{q}_j\{\phi\}_j + [K]\sum_{j=1}^{n} q_j\{\phi\}_j = F(t) \quad (2.2)$$

Multiply equation (2.2) by $\{\phi\}_i^T$ and employing the orthogonality conditions that

$$\begin{aligned} \{\phi\}_i^T[M]\{\phi\}_j &= 0 \quad \text{if } i \neq j \\ &= M_i \quad \text{if } i = j \end{aligned} \quad (2.3)$$

$$\begin{aligned} \{\phi\}_i^T[K]\{\phi\}_j &= 0 \quad \text{if } i \neq j \\ &= M_i\omega_i^2 \text{ if } i = j \end{aligned} \quad (2.4)$$

For the three mass system as shown in Fig. 4, the modal equations of motion are given by:

$$\begin{aligned} \ddot{q}_1 + C_{11}\dot{q}_1 + C_{12}\dot{q}_2 + C_{13}\dot{q}_3 + \omega_1^2 q_1 &= 0 \\ \ddot{q}_2 + C_{21}\dot{q}_1 + C_{22}\dot{q}_2 + C_{23}\dot{q}_3 + \omega_2^2 q_2 &= 0 \\ \ddot{q}_3 + C_{31}\dot{q}_1 + C_{32}\dot{q}_2 + C_{33}\dot{q}_3 + \omega_3^2 q_3 &= 0 \end{aligned} \quad (2.5)$$

It is important to note that the equations of motion expressed in terms of the generalized coordinates of undamped modes, are coupled through the modal cross-coupling damping terms C_{ij}. In the analysis of the vibrations of large structural systems, it is the normal procedure to ignore the modal cross-coupling terms.

This approximation is valid for structural systems with light damping and separation of modes. As a general rule, however, if the damping is acting at discrete locations, such as bearing or a squeeze film damper, the modal equations will not uncouple.

The condition that the modal equations uncouple is given by:

$$[C] = \alpha[M] + \beta[K] \quad (2.6)$$

Therefore, in order that the modal equations of motion uncouple, the damping matrix must be proportional to the mass or stiffness matrix. This condition is rarely encountered in actual structures in which the damping is located at only several discrete locations, as is the case with the action of bearings and seals on rotating machinery.

Example 2. Determine the modal damping coefficients for Example 1.

The damping coefficients C_1 and C_5 are assumed to be 1 lb-s/in. (175 N-s/m).

The modal damping coefficients are given by:

$$C_{ij} = \frac{1}{M_i}[C_1\phi_{i1}\phi_{j1} + C_5\phi_{i5}\bullet\phi_{j5}]$$

The modal damping matrix is given by:

$$[C] = \begin{bmatrix} 0.648 & 0 & -7.30 \\ 0 & 23.5 & 0 \\ -7.30 & 0 & 77.82 \end{bmatrix} \quad (2.7)$$

The modal equations of motion are given by equation (2.5). The modal damping matrix C is employed along with the natural frequencies as given in Table 2.1. The modal equations become

$$\ddot{q}_1 + 0.684\dot{q}_1 - 7.30\dot{q}_3 + 10{,}286 q_1 = 0$$
$$\ddot{q}_2 + 23.5\dot{q}_2 + 143{,}527 q_2 = 0 \quad (2.8)$$
$$\ddot{q}_3 - 7.30\dot{q}_1 + 77.82\dot{q}_3 + 686{,}246 q_3 = 0$$

Because of symmetry of the modes, the second mode is completely uncoupled from the first and third modes. Hence the second mode of vibration appears to act as a single degree of freedom system. The first and third modal coordinates q_1 and q_3 will be uncoupled only through the damping matrix. It will be seen that both the first and second damped eigenvalues are in considerable error for moderate values of damping and only approximately correct for small values of damping using the normal mode representation.

The characteristic polynomial generated by the normal mode method is of 6th order, whereas the characteristic polynomial for the complete system is 8th order. The second mode damped natural frequency actually increases with damping, whereas the modal equations predict the opposite trend.

3 Approximate Roots of Damped System. If the modal cross-coupling coefficients are ignored, then the equations of motion are uncoupled and are of the form

$$\ddot{q}_i + 2\xi_i\omega_i\dot{q}_i + \omega_i^2 q_i = 0 \quad (3.1)$$

Assuming q_i is of the form

$$q_i = qe^{\lambda_i t}$$

results in the following characteristics polynomial for the system:

$$\lambda_i^2 + 2\xi_i\omega_i\lambda_i + \omega_i^2 = 0 \quad (3.2)$$

where $\lambda_i = P_i \pm iv_i$

$$P = -\xi_i\omega_i \quad (3.3)$$

$$v_i = \omega_i\sqrt{1-\xi_i^2} \quad (3.4)$$

For the case where $\xi_i << 1$

$$v_i = \omega_i$$

Where the damping coefficient ξ_i is much less than 1, then the damped natural frequency v_i is equal to the undamped natural frequency, ω_i.

Note that for the single degree of freedom system the damped natural frequency reduces with increasing damping or ξ value. This *is* not necessarily the case in multidegree of freedom systems such as equation (1.9).

Table 3.1 Approximate damped eigenvalues for light damping (C_b = 1.0 lb-s/in.)(175 N-s/m)

	Mode		
	1	2	3
ω_i	101.42	378.85	828.4
ξ_i(dim)	.00337	0.03101	0.04697
$A_c = 1/(2\xi i)$	148	16	10.64
P_i	−0.342	−11.75	−38.91
v_i	101.419	378.66	827.49

The approximate damped complex eigenvalues for the system are given in Table 3.1.

In the foregoing example, the damping value of C = 1 lb-s/in. (175 N-s/m) is sufficiently low such that the damped natural frequencies correspond to the undamped natural frequencies. The values listed for mode 2 are exact for low values of damping since the second mode is uncoupled from the first and the third. It shall be seen, however, that the solution for the second mode for large values of damping is completely erroneous. As the bearing damping is increased to very large values, the natural frequencies will increase from 101.42, 378.85, and 828.4 rad/s to the constrained natural frequencies of 104.17, 413.3, and 886.9 rad/s. The approximate uncoupled modal equations predict that the damped natural frequencies decrease with increasing damping. Hence uncoupled modal equations may be completely erroneous even for moderate values of damping. This assumption of modal uncoupling is commonly used in modal analysis of structures and multilevel gas turbine engines. For the case where large localized damping forces are present in a structure or an aircraft engine, the use of uncoupled normal modal equations of motion may exhibit considerable error.

It will be seen that it is necessary to have a third order polynomial instead of a second order polynomial to obtain the characteristic behavior in which the damped natural frequency increases with damping. This is usually the case with very flexible rotors.

The quantity A_c represents the rotor modal amplification factor at an excitation frequency of $\omega = \omega_i$ and is given by:

$$Ac_i = \frac{1}{2\xi_i} = \frac{-\omega_i}{2P_i} \quad (3.5)$$

4 Solution of the Fourth Order Characteristic Polynomial Generated With Normal Modes. The coupled eigenvalue equations for the system first and third modes may be written in matrix form as follows:

$$\left[\begin{array}{c|c} \lambda^2 + C_{11}\lambda + \omega_1^2 & C_{13}\lambda \\ \hline C_{31}\lambda & \lambda^2 + C_{33}\lambda + \omega_3^2 \end{array}\right]\left[\begin{array}{c} q_1 \\ \hline q_3 \end{array}\right] = 0 \quad (4.1)$$

The characteristic polynomial for the coupled two degree of freedom system is given by expanding the determinant of the coefficient matrix as follows:

$$\lambda^4 + (C_{11} + C_{33})\lambda^3 + (C_{11}C_{33} - C_{13}C_{31} + \omega_1^2 + \omega_3^2)\lambda^2 + (C_{11}\omega_3^2 + C_{33}\omega_1^2)\lambda + \omega_1^2\omega_3^2 = 0 \quad (4.2)$$

From the knowledge of the invariants of the characteristic equation, the polynomial may be normalized in order to avoid the numerical difficulties associated with the generation of high order polynomials.

Let $\lambda = \Omega\Lambda$

where $\Omega = \sqrt{\omega_1\omega_3}$. (4.3)

The transformed polynomial equation (4.2) is now of the form:

$$\Lambda^4 + \bar{A}_3\Lambda^3 + \bar{A}_2\Lambda^2 + \bar{A}_1\Lambda + 1 = 0 \quad (4.4)$$

Note that the first and last coefficients of the transformed equation, $\bar{A}_0$ and $\bar{A}_4$ are unity.

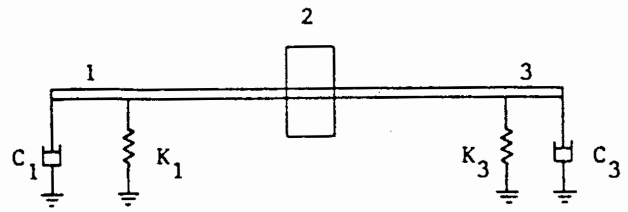

Fig. 8 Single mass system on damped flexible supports

Table 4.1 Influence of damping on system using first and third normal modes

Damping	P_1	υ_1	P_3	υ_3
0	0	101.42	0	828.4
1	−0.339	101.3	−38.9	827.44
10	−3.37	101.6	−389.1	727.6
50	−13.6	109.5	−154.4	7.4×10^{-10}
100	−12.7	120.1	−62.2	6.6×10^{-10}
1000	−1.54	126.8	−5.57	-4.49×10^{-10}

From Table 4.1, it is seen that as damping increases, from 1 lb-s/in. (175 N-s/m) to 1,000 lb-s/in. (175,000 N-s/m) at the bearings, an optimum value is reached by which maximum damping is achieved for the first mode. This damping value appears to be around $C=50$ lb-s/in. (8,750 N-s/m). However, for the third mode, it is seen that as damping increases, the damped frequency of the third mode diminishes rapidly.

From a physical standpoint, this result is not correct, as the damping increases and approaches ∞, then, the values of P_i should approach zero and the values of the damped frequencies υ_1 and υ_3 should approach the values of the constrained natural frequencies of ω_{c1} = 104.17 and ω_{c3} = 886.9 rad/s. For the case of the first mode, the asymptotic value of υ_1 approaches 126.8 rad/s rather than the value of 104.17. Hence for large values of damping, the use of normal modes is considerably in error for the third mode and only approximately accurate for the first mode. A similar problem also exists with the prediction of the second mode damped natural frequency.

The reason for this discrepancy in the calculation of the damped frequencies by the normal mode procedure is that the characteristic equation for the system is 8th order. Using three normal modes, only a 6th order system can be developed. Hence this characteristic increase in the damped frequency with increasing bearing damping cannot be predicted using only the normal modes. This situation may be remedied by the introduction of two additional rigid body modes.

However, rather than introduce the rigid body modes with normal modes, we shall examine the use of the constrained modes along with the rigid body modes.

5 Dynamical Analysis Using Constrained Normal Modes

5.1 Single-Mass System. The dynamical modal equations will be formulated using the rigid body modes $\{\phi\}_r$ and the constrained normal modes $\{\phi\}_c$. Consider the single mass system as shown in Fig. 8.

A further simplification may be obtained if symmetric bearings at station 1 and 3 are assumed. If the motion x_1 is assumed to be equal to the motion at station 3 then the equations reduce to:

$$\begin{bmatrix} 0 & \\ & M \end{bmatrix}\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} 2C & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} + \begin{bmatrix} 2K_1+K_s & -K_s \\ -K_s & K_s \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \quad (5.1.1)$$

Introducing the dimensionless time transformation τ where

$$\tau = \omega_c t;$$

$$\text{Let } f = \left(\frac{\omega_r}{\omega_c}\right)^2; \quad \bar{C} = \frac{2C}{M\omega_c}$$

The transformed equations become:

$$\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} \bar{C} & \\ & 0 \end{bmatrix}\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} + \begin{bmatrix} 1+f & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \quad (5.1.2)$$

Assume a solution of the form:

$$x = e^{\lambda t} = e^{\frac{\lambda}{\omega c}\tau} = e^{\Lambda\tau}$$

The characteristic determinant becomes:

$$\begin{bmatrix} \bar{C}\Lambda + 1 + f & -1 \\ -1 & \Lambda^2 + 1 \end{bmatrix} = 0 \quad (5.1.3)$$

The characteristic polynomial is given by:

$$\bar{C}\Lambda^3 + (1+f)\Lambda^2 + \bar{C}\Lambda + f = 0 \quad (5.1.4)$$

The modal equations will now be developed by assuming $\{x\}$ to be of the form:

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = q_r\{\phi\}_r + q_c\{\phi_c\} \quad (5.1.5)$$

where

$$\{\phi\}_r = \text{rigid body mode} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\{\phi\}_c = \text{constrained mode} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

It is of interest to note that the rigid body modes $\{\phi\}_r$ are in general not orthogonal to the constrained modes $\{\phi\}_c$.

The general equations of motion are now expressed as follows:

$$[M](\ddot{q}_r\{\phi\}_r + \ddot{q}_c\{\phi\}_c) + [C](\dot{q}_r\{\phi\}_r + \dot{q}_c\{\phi\}_c) + [K](q_r\{\phi\}_r + q_c\{\phi\}_c) = 0 \quad (5.1.6)$$

Multiplying equation (5.1.2) by $\{\phi_r\}^T$ and $\{\phi\}_c^T$ yields the following matrix equations:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \ddot{q}_r \\ \ddot{q}_c \end{bmatrix} + \begin{bmatrix} \bar{C} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_r \\ \dot{q}_c \end{bmatrix} + \begin{bmatrix} f & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q_r \\ q_c \end{bmatrix} = 0 \quad (5.1.7)$$

The characteristic determinant is given by:

$$\begin{bmatrix} \Lambda^2 + \bar{C}\Lambda + f & \Lambda^2 \\ \Lambda^2 & \Lambda^2 + 1 \end{bmatrix}\begin{bmatrix} q_r \\ q_c \end{bmatrix} = 0 \quad (5.1.8)$$

The characteristic polynomial is:

$$(\Lambda^2 + \bar{C}\Lambda + f)(\Lambda^2 + 1) - \Lambda^4 = 0$$

Expanding the above, we obtain the following:

$$\bar{C}\Lambda^3 + (1+f)\Lambda^2 + \bar{C}\Lambda + f = 0 \quad (5.1.9)$$

From the comparison of the polynomial generated by the exact system and the constrained modal analysis it is seen that the polynomial is exact.

From equation (5.1.4) it is seen that as $\bar{C}-0$ the characteristic equation becomes:

$$(1+f)\Lambda^2 + f = 0 \quad (5.1.10)$$

The natural frequency of the system is given by:

$$\Lambda = \pm i\sqrt{\frac{f}{1+f}}$$

or

$$\omega_1 = \omega_c \Lambda = \sqrt{\frac{f}{1+f}}\sqrt{\frac{K_s}{M}} \qquad (5.1.11)$$

The normal mode shape is given by:

$$\{\phi\} = \begin{bmatrix} \frac{1}{1+f} \\ 1 \end{bmatrix} \qquad (5.1.12)$$

Using the normal mode formulation the modal equation becomes:

$$\Lambda^2 + \left(\frac{1}{1+f}\right)^2 \bar{C}\Lambda + \frac{f}{1+f} = 0 \qquad (5.1.13)$$

By comparing the approximate formulation with the exact third order polynomial it is seen that the approximate solution is accurate only when C is small and for $f << 1$. These two conditions imply light damping and stiff rotors or shafts. Thus in the case of analysis of highly flexible rotor bearing systems with substantial damping, the normal modal method may be inaccurate for even moderately damped bearings.

Example 3. Determine the characteristic equations and the first damped mode of the three mass rotor for various damping values using constrained modes.

Let

$$\{x\} = q_r\{\phi_r\} + q_c\{\phi_c\}$$

where

$$\phi_r^T = [1\ 1\ 1\ 1\ 1]$$
$$\phi_c^T = [0\ 0.7071\ 1\ 0.7071\ 0]$$

The modal equations of motion are given by:

$$\begin{bmatrix} M_{rr} M_{cr} \\ M_{rc} M_{cc} \end{bmatrix}\begin{bmatrix} \ddot{q}_r \\ \ddot{q}_c \end{bmatrix} + \begin{bmatrix} 2C & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_r \\ \dot{q}_c \end{bmatrix} + \begin{bmatrix} \omega_r^2 M_{rr} & 0 \\ 0 & M_{cc}\omega_c^2 \end{bmatrix}\begin{bmatrix} q_r \\ q_c \end{bmatrix} = 0 \qquad (5.1.14)$$

where

$$M_{ij} = \phi_i^T[M]\phi_j$$

The resulting characteristic equation for the system is given by:

$$(1 - \bar{M}_{cr}\bar{M}_{rc})\Lambda^4 + \bar{C}\Lambda^3 + (1+f)\Lambda^2 + \bar{C}\Lambda + f = 0 \qquad (5.1.15)$$

where

$$\bar{C} = \frac{2C}{M_{rr}\omega_c}, \quad \bar{M}_{cr} = \frac{M_{cr}}{M_{rr}}$$

$$f = \frac{\omega_r^2}{\omega_c^2}, \quad \bar{M}_{rc} = \frac{M_{rc}}{M_{cc}}$$

Thus, for the three mass system:

$M_{rr} = 0.01395$, $M_{rc} = M_{cr} = 0.01123$; $M_{cc} = M_1 = 0.0095$

$$\omega_r = \sqrt{\frac{2K_b}{3m}} = 378.6 \text{ rad/s}; \ \omega_c = 104.17 \text{ rad/s}$$

$$\bar{C} = \bar{C}_{rr} = \frac{\phi_r^T[C]\phi_r}{M_{rr}\omega_c} = 1.376C;$$

$$f = \left(\frac{\omega_r}{\omega_c}\right)^2 = \left(\frac{378.6}{104.17}\right)^2 = 13.2$$

Table 5.1 Damped first mode for various values of C-constrained modes and exact solution

	Constrained			Exact	
C	P_1	ν_1	A_c	P_1	ν_1
0.1	−0.034	100.59	2958		
1.0	−0.339	100.63	147.5	−0.339	100.4
5	−1.414	101.00	35.7	−1.429	101.0
7	−1.684	101.70	30.2	−1.709	101.5
10	−1.817	102.34	28.2	−1.84	102.1
12	−1.794	102.67	28.6	−1.82	102.5
20	−1.468	103.4	35.2	−1.48	103.3
100	−0.356	104.14	146.26	−0.363	104.0

The characteristic polynomial is given by

$$0.04874\Lambda^4 + 1.376C\Lambda^3 + 14.209\Lambda^2 + 1.376C\Lambda + 13.209 = 0 \qquad (5.1.16)$$

The primary complex root of this equation for various values of damping is given in Table 5.1.

From Table 5.1 it is seen that the system first damped natural frequency ν_1 increases in magnitude from the value of the undamped natural frequency ω_1 of 100.6 rad/s to the constrained natural frequency of ω_{c1} = 104.14 rad/s as the bearing damping C increases. It is apparent also that there is an optimum bearing damping that should be placed at the supports and this value is approximately $C = 10$ lb-s/in. (1750 N-s/m) and not 50 lb-s/in. (8,750 N-s/m) as indicated in Table 4.1. The minimum amplification factor that can be achieved with this system with optimum damping is only $A_c = 28$. Hence it is apparent that the first mode of the 3 mass system is extremely sensitive to excitation and will have a large amplitude of motion at the first critical speed due to rotor unbalance.

In comparison of the results of Table 5.1 to the results as shown in Table 4.1 using normal modes, it is seen that the prediction of the first mode root for C = 1 lb-s/in. (175 N-s/m) yields a real component of P_1 = −0.339 rad/s. For C = 10 lb-s/in. (1750 N-s/m), the value of P_1 as shown in Table 4.1 is −3.37 rad/s which is almost twice as large as the value shown in Table 5.1. Hence it can be concluded that the employment of normal modes in even moderately damped systems may lead to sizable errors in the prediction of the value of the real root.

5.2 Three-Mass Rotor. In Section 5.1 it was seen for the single-mass case that the employment of a constrained mode $\{\phi\}_c$ in conjunction with a rigid body mode $\{\phi\}_r$ generated the correct third order characteristic polynomial. This procedure, therefore will be applied to the three-mass, five-station system. The system displacements will be assumed to be imposed of three constrained modes plus the addition of two rigid body modes. The displacements will be given by:

$$\{X\} = \sum_{i=1}^{3} q_{ic}\{\phi\}_c + \sum_{i=1}^{2} q_{ir}\{\phi\}_r \qquad (5.2.1)$$

It is important to note that the constrained modes are not necessarily orthogonal to the rigid body modes.

The displacements $\{X\}$ may be expressed in terms of either the displacement mode shapes $\{\phi\}$ or in terms of the orthonormal mode shapes $\{\Phi\}$ where

$$\{\Phi\}_i = \frac{1}{\sqrt{Mi}}\{\phi\}_i \qquad (5.2.2)$$

Table 5.2 represents the five orthonormal mode shapes required to describe the three-mass system of Fig. 4. The first three mode shapes listed in Table 5.2 are constrained modes and the last two are rigid body cylindrical and conical modes.

The displacements can be written in general as:

Table 5.2 Orthonormal mode shapes and eigenvalues for three-mass system

	Constrained modes			Rigid body	
Mode no.	1	2	3	4	5
Freq. r/min	995	3,947	8,470	0	0
rad/s	104.17	413.3	886.9		
M-modal mass (lb-s^2/in.)	0.0093	0.0093	0.0093	0.01395	.0093
Station 1	0	0	0	8.467	20.739
2	7.332	10.370	−7.332	8.467	10.370
3	10.370	0.0	10.370	8.467	0.0
4	7.332	−10.370	−7.332	8.467	−10.370
5	0	0	0	8.467	−20.739

$$\{X\} = \sum_{i=1}^{5} q_i \{\Phi\}_i \tag{5.2.3}$$

Applying the foregoing displacement relationship to equation (1.9) and multiplying by $\{\Phi\}^T$ results in the following set of modal equations for the generalized coordinates q_i:

$$\begin{aligned}
\ddot{q}_1 + M_{14}\ddot{q}_4 + M_{15}\ddot{q}_5 + \omega_{c1}{}^2 &= 0\\
\ddot{q}_2 + M_{24}\ddot{q}_4 + M_{25}\ddot{q}_5 + \omega_{c2}{}^2 &= 0\\
\ddot{q}_3 + M_{34}\ddot{q}_4 + M_{35}\ddot{q}_5 + \omega_{c3}{}^2 &= 0\\
\ddot{q}_4 + M_{41}\ddot{q}_1 + M_{42}\ddot{q}_2 + M_{43}\ddot{q}_3 + C_{44}\dot{q}_4 + \omega_{r1}{}^2 &= 0\\
\ddot{q}_5 + M_{51}\ddot{q}_1 + M_{52}\ddot{q}_2 + M_{53}\ddot{q}_3 + C_{55}\dot{q}_5 + \omega_{r2}{}^2 &= 0
\end{aligned} \tag{5.2.4}$$

It is seen that the mass system is not diagonal. It is given by:

$$[M] = \begin{bmatrix} 1.00 & 0 & 0 & 0.9856 & 0 \\ 0 & 1.00 & 0 & 0 & 1.00 \\ 0 & 0 & 1.00 & -0.1691 & 0 \\ 0.9856 & 0 & -0.1691 & 1 & 0 \\ 0 & 1.00 & 0 & 0 & 1 \end{bmatrix} \tag{5.2.5}$$

The coefficients for the damping matrix are given by:

$$[C] = C\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 143.38 & \\ & & & & 215 \end{bmatrix} \tag{5.2.6}$$

The total stiffness matrix of the system is composed of the bearing stiffness matrix plus the stiffness matrix of the shaft corresponding to free-free end conditions

$$[K] = [K]_b + [K]_S \tag{5.2.7}$$

The constrained frequencies are given by:

$$\begin{aligned}
\omega_{ci}{}^2 &= \{\Phi\}_{ci}{}^T[[K]_b + [K]_S]\{\Phi\}_{ci}\\
&= \{\Phi\}_{ci}{}^T[K]_S\{\Phi\}_{ci}
\end{aligned} \tag{5.2.8}$$

In the absence of external bearings acting at the interior points X_2 to X_4. The components $\omega_{ci}{}^2$ are given by:

$$\omega_{c1}{}^2 = 104.17^2 = 10{,}851.4$$

$$\omega_{c2}{}^2 = 413.3^2 = 170{,}816.9$$

$$\omega_{c3}{}^2 = 886.9^2 = 786.591.6$$

The rigid body frequencies are given by:

$$\begin{aligned}
\omega_{ri}{}^2 &= \{\Phi\}_{ri}{}^T[[K]_b + [K]_S]\{\Phi\}_{ri}\\
&= \{\Phi\}_{ri}{}^T[K]_b\{\Phi\}_{ri}
\end{aligned} \tag{5.2.9}$$

Since $[K]_S$ is a singular matrix of order 2,

$$\omega_{r1}{}^2 = \{\phi\}_{r1}{}^T[K]_b\{\phi\}_{r1} = 143{,}380 = \frac{2K_b}{3m}$$

$$\omega_{r2}{}^2 = \{\Phi\}_{r2}{}^T[K]_b\{\Phi\}_{r2} = 860{,}212 = \frac{4K_b}{m} \tag{5.2.10}$$

The modal stiffness matrix is given by:

$$[K]_{\text{modal}} = 10^5\begin{bmatrix} 0.108514 & 0 & 0 & 0 & 0 \\ 0 & 1.7082 & 0 & 0 & 0 \\ 0 & 0 & 7.8659 & 0 & 0 \\ 0 & 0 & 0 & 1.433 & 0 \\ 0 & 0 & 0 & 0 & 8.6021 \end{bmatrix} \tag{5.2.11}$$

The diagonal modal $[K]$ matrix may be normalized by dividing the elements by $\omega_{c1}{}^2$.

$$[\bar{K}]_{\text{modal}} = \begin{bmatrix} 1 & & & & \\ & \bar{f}_2 & & & \\ & & \bar{f}_3 & & \\ & & & \bar{f}_4 & \\ & & & & \bar{f}_5 \end{bmatrix} \tag{5.2.12}$$

where

$$\bar{f}_i = \left(\frac{\omega_i}{\omega_{c1}}\right)^2 \tag{5.2.13}$$

The diagonal damping matrix is normalized by dividing by ω_{c1}. This normalization procedure is equivalent to a dimensionless time transformation of the form $\omega_{c1}t = \tau$. The normalized $[C]$ matrix is given by:

$$[\bar{C}] = C\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1.376 & \\ & & & & 2.065 \end{bmatrix} \tag{5.2.14}$$

The system modal equations of motion are given by:

$$[\bar{M}]\{\ddot{q}\} + [\bar{C}]\{\dot{q}\} + [\bar{f}_i]\{q\} = 0 \tag{5.2.15}$$

6 Generation of Characteristic Equation. The system given as follows,

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = F(t) \tag{6.1}$$

may be converted into a system of coupled first order equations by:

$$\{y\} = \begin{bmatrix} \{\dot{x}\} \\ \{x\} \end{bmatrix}; \quad \{\dot{y}\} = \begin{bmatrix} \{\ddot{x}\} \\ \{\dot{x}\} \end{bmatrix}$$

$$\begin{bmatrix} 0 & [M] \\ [M] & [C] \end{bmatrix}\begin{bmatrix} \{\ddot{x}\} \\ \{\dot{x}\} \end{bmatrix} + \begin{bmatrix} [-M] & 0 \\ 0 & [K] \end{bmatrix}\begin{bmatrix} \{\dot{x}\} \\ \{x\} \end{bmatrix} = \begin{bmatrix} 0 \\ \{F(t)\} \end{bmatrix} \tag{6.2}$$

The form of the $2n$ order equation in $\{y\}$ is given by:

$$[A]\{\dot{y}\} + [B]\{y\} = \{Q\} \tag{6.3}$$

Consider the homogeneous equation with $\{Q\} = 0$. Let

$$\{y\} = \{Y\}e^{st}$$

$$[D]\{y\} = \lambda\{Y\} \quad (6.4)$$

where $\lambda = \dfrac{1}{s}$ inverse complex root

$$[D] = -[B]^{-1}[A]$$

$$= -\begin{bmatrix} -[M]^{-1} & 0 \\ 0 & [K]^{-1} \end{bmatrix} \begin{bmatrix} 0 & [M] \\ [M] & [C] \end{bmatrix} \quad (6.5)$$

$$= \left[\begin{array}{c|c} 0 & I \\ \hline -[K]^{-1}[M] & -[K]^{-1}[C] \end{array}\right] \quad (6.6)$$

The foregoing system may be iterated directly to determine the complex eigenvalues or the characteristic polynomial may be expanded by the application of Leverrier's algorithm. The characteristic polynomial is given by:

$$[D] - \lambda[I] = 0 \quad (6.7)$$

If the foregoing matrices were used in Leverrier's algorithm, then considerable numerical difficulties would result. The $K^{-1}M$ and $K^{-1}C$ matrices may be scaled as follows:
Let

$\Omega t = \tau$

$$\Omega^2[M]\{\ddot{x}\} + \Omega[C]\{\dot{x}\} + [K]\{x\} = 0 \quad (6.8)$$

$$\Omega^2[K]^{-1}[M]\{\ddot{x}\} + \Omega[K]^{-1}[C]\{\dot{x}\} + \{x\} = 0 \quad (6.9)$$

Let the choice of Ω be the first constrained natural frequency (104 rad/s).

The normalized matrices are given by:

$$[K]^{-1}[M] = \begin{bmatrix} 0.0000 & 0.0377 & 0.0251 & 0.0126 & 0.0000 \\ 0.0000 & 0.3163 & 0.3739 & 0.2409 & 0.0000 \\ 0.0000 & 0.3739 & 0.5321 & 0.3739 & 0.0000 \\ 0.0000 & 0.2409 & 0.3739 & 0.3163 & 0.0000 \\ 0.0000 & 0.0126 & 0.0251 & 0.0377 & 0.0000 \end{bmatrix}$$

$$[K]^{-1}[C] = \begin{bmatrix} 1.0400 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.7800 & 0.0000 & 0.0000 & 0.0000 & 0.2600 \\ 0.5200 & 0.0000 & 0.0000 & 0.0000 & 0.5200 \\ 0.2600 & 0.0000 & 0.0000 & 0.0000 & 0.7800 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0400 \end{bmatrix}$$

The assembled dynamic matrix D is given by

$$[D] = \left[\begin{array}{ccccc|ccccc} 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \\ \hline 0.0000 & -0.0377 & -0.0251 & -0.0126 & 0.0000 & -1.0400 & 0.0000 & 0.0000 & 0.0000 & -0.0000 \\ 0.0000 & -0.3163 & -0.3739 & -0.2409 & 0.0000 & -.7800 & 0.0000 & 0.0000 & 0.0000 & -0.2600 \\ 0.0000 & -0.3739 & -0.5321 & -0.3739 & 0.0000 & -.5200 & 0.0000 & 0.0000 & 0.0000 & -0.5200 \\ 0.0000 & -0.2409 & -0.3739 & -0.3163 & 0.0000 & -.2600 & 0.0000 & 0.0000 & 0.0000 & -0.7800 \\ 0.0000 & -0.0126 & -0.0251 & -0.0377 & 0.0000 & -.0000 & 0.0000 & 0.0000 & 0.0000 & -1.0400 \end{array}\right]$$

Let the characteristic equation be expressed in the form:

$$\wedge^n + A_1\wedge^{n-1} + A_2\wedge^{n-2} + A_n = 0 \quad (6.10)$$

The coefficients of the characteristic equation may be determined by Leverrier's algorithm as follows:

$$A_1 = -\text{trace}\,[D] \quad (6.11)$$

$$[B]_1 = [D] + A_1[I]$$

$$A_2 = -\tfrac{1}{2}\,\text{trace}\,[[D][B]_1]$$

or in general,

$$A_k = -\frac{1}{k}\,\text{trace}[[D][B_{k-1}]];\ k > 1 \quad (6.12)$$

and

$$[B]_k = [D][B_{k-1}] + A_k[I] \quad (6.13)$$

Using Leverrier's algorithm, an eighth order polynomial is generated. Table 6.1 represents the damped roots for the three mass system with the damping varied from 0.1 to 100 lb-s/in. The roots generated by expanding out the complete M, K, and C matrices were identical (within numerical bounds) to the roots generated based on the use of constrained and rigid body modes. Since the five modes used form a complete set, the exact characteristic polynomial is generated. The exact solution for the first mode was generated also by using only the first constrained and the first rigid body mode. In comparison with Table 4.1, the first damped mode based on the undamped normal modes is only accurate for small ranges of damping and the third mode is completely in error.

From Table 6.1 it is of interest to note that as the damping is increased, the damped natural frequencies increase from the undamped normal mode and approach the constrained normal mode values. It is also of importance to note that the optimum damping for all three modes is not identical. This fact is important when designing a squeeze film bearing for a gas turbine which must operate through multiple modes. For example, with the three-mass test rotor, the optimum damping for the second and third modes is between 1 to 2 lb-s/in. (175 to 350 N-s/m) damping, while the optimum damping for the first mode is 10 lb-s/in. (1750 N-s/m). Even with the inclusion of optimum damping for the first mode, the lowest possible amplification factor is given as 27.7.

III Summary and Conclusions

(1) There are three sets of undamped modes of motion that may be used as building blocks or modal sets to determine the system damped eigenvalues or forced response. These modes are called the normal modes, the constrained normal modes, and the rigid body free-free modes.

(2) The normal mode set yields only the correct damped eigenvalues for low values of damping. The free-free mode set should be avoided. These particular modes do not form a complete set of functions to properly span the vector space.

(3) The undamped normal modes are generated by setting the damping equal to zero. When these modes are used to express the dynamical equations of motion, the existence of damping caused by bearings or seals will cause modal cross-coupling damping terms to appear which couples the equations of motion. The normal structural procedure is to ignore the modal cross-coupling damping terms by assuming that the damping matrix is proportional to the mass or the stiffness matrix. With real turbomachinery, such a condition never occurs and the equations of motion cannot be considered to be uncoupled unless the modal damping coefficients are extremely low, of the order of two percent of critical damping.

(4) For the case of moderate to high values of bearing damping, the use of the normal modes will not result in the

Table 6.1 Damped roots of three-mass rotor systems

C	P_1	V_1	A_1	P_2	V_2	A_2	P_3	V_3	A_3
0.1	−.03	100.4	1453.0	−1.19	387.7	158.6	−4.05	828.3	102.4
1.0	−.342	100.4	146.6	−11.00	382.0	17.3	−28.4	847.4	14.9
2.0	−.67	100.5	75.1	−17.51	389.8	11.1	−28.02	868.1	15.5
5.0	−1.43	101.0	35.4	−15.98	406.4	12.7	−14.79	882.7	29.8
7.0	−1.71	101.5	29.8	−12.71	410.1	16.1	−10.87	884.5	40.7
10.0	−1.84	102.1	27.7	−9.44	412.4	21.8	−7.72	885.4	57.3
12.0	−1.82	102.5	28.2	−8.00	413.1	25.8	−6.46	885.7	68.5
15.0	−1.71	102.9	30.1	−6.50	413.7	31.8	−5.19	886.0	85.4
20.0	−1.48	103.3	34.9	−4.93	414.2*	42.0	−3.90	886.2	113.5
50.0	−.71	103.9	73.5	−2.00	414.7*	103.9	−1.55	886.4	285.7
100.0	−.36	104.0	143.2	−.89	413.7	231.7	*	*	*

*Numerical inaccuracy in root.

correct eigenvalues, either as to the real part (the damping) or the imaginary part (the damped natural frequency).

The prediction of the first damped mode is only moderately accurate, using the normal modes. However, the second and third damped modes are considerably in error for large values of bearing damping. In previous work presented by Li and Gunter, on analysis of gas turbine engine vibrations by the normal mode approach, many higher order modes had to be retained in order to maintain accuracy of the lower mode response.

(5) There is only one set of modes that was found to generate the exact characteristic polynomial. This was the use of the three constrained flexible modes, plus the addition of two rigid body modes. This particular modal set has the advantage that the constrained normal modes are obtained by specifying 0 bearing displacements. Therefore the constrained modes are independent of bearing stiffness. The characteristic polynomial for the second mode, for example, is a third order polynomial instead of a second order, as would be obtained from the normal mode approach. The use of the second constrained mode and the second or conical rigid body mode results in the exact third order characteristic polynomial. The behavior of the third order characteristic polynomial is completely different from the second order. For example, with a second order polynomial, the damped natural frequency decreases with increasing damping. However, the opposite effect occurs with the third order polynomial. The frequency may increase with increasing damping. As the damping is increased to high values, there is an asymptotic value that is reached which represents the constrained critical speed (pinned ends).

The first, third, and cylindrical rigid body modes combine to form a fifth order characteristic polynomial. The correct behavior of both the first and third modes are generated by this set of modes. It is also of interest to note that the exact values of the first mode were generated by using only the first constrained and cylindrical rigid body mode.

(6) In this paper, it was shown how the exact characteristic polynomial may be generated by the use of Leverrier's algorithm. By means of this algorithm, the coefficients of the characteristic polynomial may be rapidly determined. These characteristic coefficients may be examined by Routh's criteria for stability, or the characteristic equation may be solved directly. Normally, one should not attempt to generate the characteristic polynomial for large order systems, as the coefficients of the polynomial become increasingly large. In this paper, it is shown how a simple scaling procedure may be incorporated with the generation of the characteristic polynomial in order to keep the coefficients within bounds.

The scaling procedure was found to be most successful and coefficients for twentieth-order polynomials can be easily generated. This procedure has been adapted to the general stability analysis of turborotors, including eight bearing stiffness and damping coefficients per bearing and shaft gyroscopics. The application of this method to the analysis of at 70 MW gas turbine-generator will be presented in detail at a later date.

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APPENDIX

Nomenclature-Appendix

$[I_p]$ $= n \times n$ rotor polar moment of inertia matrix
$[I_t]$ $= n \times n$ rotor transverse moment of inertia matrix
$[G]$ $= n \times n$ skew symmetric gyroscopic matrix
$\{U\}^T$ $= [X\,\theta\,Y\,\psi]^T$

A. Generalized Constrained Modal Equations of Motion

In an actual turborotor, the equations of motion are more complex than the simple three-mass system, as represented by equation (1.9). The presence of gyroscopic disk moments, fluid film bearings, and seals causes the rotor equations in the horizontal and vertical directions to be cross coupled. The general linearized equations of motion are of the form (A.1).

$$\begin{bmatrix} M & 0 & 0 & 0 \\ 0 & I_t & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & I_t \end{bmatrix}_M \begin{Bmatrix} \{\ddot X\} \\ \{\ddot\theta\} \\ \{\ddot Y\} \\ \{\ddot\psi\} \end{Bmatrix} + \Omega \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_p \\ 0 & 0 & 0 & 0 \\ 0 & -I_p & 0 & 0 \end{bmatrix}_G \begin{Bmatrix} \{\dot X\} \\ \{\dot\theta\} \\ \{\dot Y\} \\ \{\dot\psi\} \end{Bmatrix} + \begin{bmatrix} C_{xx} & & C_{xy} & \\ & C_{\theta\theta} & & C_{\theta\psi} \\ C_{yx} & & C_{yy} & \\ & C_{\psi\theta} & & C_{\psi\psi} \end{bmatrix}_C \begin{Bmatrix} \{\dot X\} \\ \{\dot\theta\} \\ \{\dot Y\} \\ \{\dot\psi\} \end{Bmatrix} + \begin{bmatrix} K_{rr} & K_{rs} & 0 & 0 \\ K_{rs}^T & K_{ss} & 0 & 0 \\ 0 & 0 & K_{rr} & K_{rs} \\ 0 & 0 & K_{rs}^T & K_{ss} \end{bmatrix}_S \begin{Bmatrix} \{X\} \\ \{\theta\} \\ \{Y\} \\ \{\psi\} \end{Bmatrix} + \begin{bmatrix} K_{xx} & & K_{xy} & \\ & K_{\theta\theta} & & K_{\theta\psi} \\ K_{yx} & & K_{yy} & \\ & K_{\psi\theta} & & K_{\psi\psi} \end{bmatrix}_B \begin{Bmatrix} \{X\} \\ \{\theta\} \\ \{Y\} \\ \{\psi\} \end{Bmatrix} = 0 \quad (A.1)$$

X and Y represent the displacements measured in a fixed X, Z, and Y, Z coordinate system and θ, ψ are the corresponding rotations measured respectively in the fixed reference system. The equations of motion are cross coupled due to the gyroscopic G matrix and the bearing matrix B. Self-excited instability is caused primarily by the cross coupling bearing coefficients.

In matrix notation similar to equation (1.1), the complete equations of motion may be expressed as:

$$[M]\{\ddot U\} + ([G] + [C])\{\dot U\} + ([K]_s + [K]_b)\{U\} = 0.$$

The generation of the modal equations of motion is as follows. Let

$$\begin{Bmatrix} X \\ \theta \end{Bmatrix} = \sum_{c=1}^{n} q_{xc}\{\phi_c\} + \sum_{r=1}^{2} q_{xr}\{\phi_r\}$$
$$\begin{Bmatrix} Y \\ \psi \end{Bmatrix} = \sum_{c=1}^{n} q_{yc}\{\phi_c\} + \sum_{r=1}^{2} q_{yr}\{\phi_r\} \quad (A.3)$$

The modal equations are of the form:

$$\begin{bmatrix} \ddots 1 & 0 & M_{rc} & 0 \\ 0 & 1 \ddots & 0 & M_{rc} \\ M_{rc} & 0 & M & 0 \\ 0 & M_{rc} & 0 & M \end{bmatrix} \begin{Bmatrix} \{\ddot q_{cx}\} \\ \{\ddot q_{cy}\} \\ \{\ddot q_{rx}\} \\ \{\ddot q_{ry}\} \end{Bmatrix} + \begin{bmatrix} 0 & \omega I_p & 0 & 0 \\ -\omega I_p & 0 & 0 & 0 \\ 0 & 0 & \bar C_{xx} & \bar C_{xy} \\ 0 & 0 & \bar C_{yx} & \bar C_{yy} \end{bmatrix} \begin{Bmatrix} \{\dot q_{cx}\} \\ \{\dot q_{cy}\} \\ \{\dot q_{rx}\} \\ \{\dot q_{ry}\} \end{Bmatrix} + \begin{bmatrix} \ddots\omega_i^2 & & 0 & 0 \\ & \omega_i^2\ddots & 0 & 0 \\ 0 & 0 & \bar K_{xy} & \bar K_{xy} \\ 0 & 0 & \bar K_{yx} & \bar K_{yy} \end{bmatrix} \begin{Bmatrix} \{q_{cx}\} \\ \{q_{cy}\} \\ \{q_{rx}\} \\ \{q_{ry}\} \end{Bmatrix} = 0 \quad (A.4)$$

B. Routh-Hurwitz Stability Criterion

The general characteristic polynomial, using two rigid modes and C constrained modes, is of order $N = 8 + 4C$. If the characteristic polynomial is expressed in the form

$$P(\lambda) = \sum_{k=0}^{n} A_{n-k}\lambda^k = 0, \quad (B.1)$$

then the general Routh-Hurwitz criterion may be used to determine the stability of the system. A stable system implies that all of the complex roots have real negative roots. The first necessary, but not sufficient, condition is that

$$A_i > 0. \quad (B.2)$$

Thus, the Routh array is formed:

$$D = \begin{vmatrix} A_1 & A_0 & 0 & 0 & \cdots \\ A_3 & A_2 & A_1 & A_0 & \cdots \\ A_5 & A_4 & A_3 & A_2 & \cdots \\ A_7 & A_6 & A_5 & A_4 & \cdots \end{vmatrix} \quad D_0,\ D_1,\ D_2,\ D_3,\ \ldots,\ D_n \quad (B.3)$$

The condition of stability is that all of the D_n determinants must be positive. For the study of turborotor stability, it is usually only necessary to retain the first constrained planar mode and the two rigid body modes.

DISCUSSION

H. A. Scarton.[1] To those of us who try to remain independent of large computing packages, such as NASTRAN, IMSL or SSP subroutines, the vastly simplified results given by this work come as a welcome relief. A few questions relating to certain details of this major contribution remain.

First of all, could the authors comment on the extension of this problem to the nonself-adjoint (nonsymmetrical **M**, **C**, **K** matrices) problem? A casual analysis of this formalism leading up to equations (6.10-13) suggests that there may be little modification required. Second, can the authors elaborate, in detail on their procedure for handling the case of either singular **K** (rigid body mode) or **M** (no mass at certain system connection points)? Third, would the authors care to generalize their λ scaling criterion? In one instance, they normalize by the lowest constrained mode; in another they use the geometric mean of the lowest and the highest modes (by which I assume they mean the damped vibration mode wherein the coupling damping terms are ignored). What about the linear average of the lowest and the highest? Further, do the authors suggest that one must first solve the conservative problem to obtain these normalization factors? In that instance, how would they recommend that the normalization factors for the conservative problem be obtained?

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Authors' Closure

The following discussion is in reply to the various components put forth by Dr. Scarton concerning several aspects of this paper.

His first point dealt with the question of how this work may be applied to problems with nonself-adjoint **M**, **C**, and **K** matrices. This is of major importance to those, for example, who wish to investigate problems of self-excited rotor instability. The mass matrix is, in general, always symmetric, but the **C** and **K** matrices are not symmetric for the case of turborotors with gyroscopic effects, fluid film bearings, or aerodynamic cross-coupling forces. The gyroscopic moments cause skew-symmetric terms to appear in the general damping matrix, and the fluid film bearings cause asymmetrical terms to appear in the general stiffness matrix. For the case of symmetric **M**, **C**, and **K** matrices, the complex eigenvalues all have negative real components (stable system). The generalized asymmetric **K** matrix may lead to the existence of complex roots with positive real roots. In this case the system is said to be unstable in the linear sense. The existence of a real positive root in the system may be determined by the application of Rouths' criterion on the coefficients of the characteristic equations.

In the simple example of the three-mass rotor as represented by equation (1.9), the motion is planar and all of the matrices are symmetric. Hence all the complex roots will be stable. In order to examine the case of nonself-adjoint systems, the order of the matrices is doubled to a $2n \times 2n$ system as given in Appendix A. The stability of the system may be determined by the Routh-Hurwitz Stability Criterion as presented in Appendix B.

The second point is related to the handling of special cases in which **M** or **K** are singular. The case in which **M** is singular represents no problem. For example, in the three-mass, five-station model of equation (1.9), the mass matrix is singular since zero mass was assumed to be present at the first and fifth stations. In the development of the $2n \times 2n$ dynamical **D** matrix as given in equation (6.6), the mass matrix is not inverted. In the present formulation, difficulty would be encountered if the **K** matrix were singular. This would correspond to the case of zero-bearing coefficients and a rotor or shaft spinning freely in space. This problem, however, does occur when one wishes to calculate the shaft free-free bending modes. The problem of the singular **K** matrix may be overcome by reformulating the dynamical **D** matrix to require the inversions of the mass matrix instead of the **K** matrix. If, however, both **K** and **M** are singular, which is often the case for free-free eigenvalue analysis, the system may be modified by adding small bearing stiffness values to the **K** matrix to make it nonsingular. With proper scaling, this method is numerically stable. The first two modes are the damped rigid body modes, and the higher modes are the free-free modes.

The third important point that Dr. Scarton makes concerns the general procedure for scaling the system by selecting the proper λ value. There are several ways to theoretically select the optimum λ value. It should be reemphasized, that with no scaling, numerical difficulty is experienced with even relatively simple systems. The ideal scaling is achieved by equation (4.4) in which the A_n coefficient is set equal to unity. For example, for the fourth-order system as represented by equation (4.2), the scaling factor $\Omega = \sqrt{\omega_1 \omega_3} = A_n^{1/4}$.

An accurate scaling factor may be determined from using:

$$\lambda = \mathrm{DET}\,|[\mathbf{M}^{-1}][\mathbf{K}]|^{1/N}$$

or if **M** is singular,

$$\lambda^{-1} = \mathrm{DET}\,|[\mathbf{M}][\mathbf{K}]^{-1}|^{1/N}$$

It is also important to note that since this method was implemented on a microcomputer (both the HP-9845 and HP-200 with 32-bit internal architecture), various scale factors could be quickly tested. Any reasonable estimate of a scaling factor produced good results. The problem can also be formulated whereby the polynomial can be truncated above a certain order, dropping the higher frequency modes, which usually are not of interest in a standard vibration problem.

It is important to note that this procedure has a wide range of application beyond rotating shafts, and has also been used to formulate the damped torsional eigenvalues of complex gear boxes.