DYNAMICS OF MULTI-SPOOL GAS TURBINES USING THE MATRIX TRANSFER METHOD - THEORY

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Abstract. *This paper presents a theoretical procedure for the dynamic analysis of multi-spool turborotors with flexible supports and flexible branches using a modified transfer matrix method in which multi-span rotors with multiple branches may be computed accurately and rapidly on a small engineering workstation.*

Disk and shaft gyroscopic moments are included in order to compute synchronous, forward and backward critical speeds. The advantages of the transfer matrix method are that elaborate models may be easily set up, and eigenvalues and eigenvectors (frequencies and mode shapes) may be quickly determined. From the relative modes, the kinetic and strain energy distribution along the span and elements may be determined. The knowledge of the energy distribution for each mode greatly assists in rotor-bearing design and optimization.

In order to produce accurate eigenvalues and mode shapes by the transfer matrix method, scaling must be incorporated into the transfer matrix equations. In this paper, a unique scaling procedure has been introduced into the transfer matrices by appropriate transformations of slope, moment, and shear coefficients.

Also the numerical difficulties caused by branches are clearly described using a simple model, and a method has been developed to remove this problem. A modified determinant search procedure has been developed in which singular branch roots are factored out of the characteristic polynomial.

The computer algorithm used to calculate undamped critical speeds of multi-span rotors with multiple branches is presented in this paper. This algorithm is designed to handle variable station length, multiple rotor spans with multiple branches between spans, strain and kinetic energy distribution, and graphic display of the calculated mode shapes.

Keywords. *Critical speeds, mode shapes, transfer matrix analysis, multi-spool rotors, branched rotor systems.*

1. INTRODUCTION

With modern turbo-rotors, such as large steam turbines on flexible foundations, and aircraft engines, the rotor dynamics must consider the influence of flexible supports and flexible casings. For example, with a complex commercial gas-turbine engine, as many as five levels or rotor spans must be used to represent the low speed fan rotor, the high speed turbine-compressor, and the various casing and diffuser sections.

An analysis of this complexity normally requires a finite element procedure on a large mainframe computer. The finite element procedures are ideally suited for handling multi-level rotors with branches, and interconnecting or differential bearings. However, a high speed mainframe computer is normally required to solve the eigenvalues of such a large system. In addition, many of the current finite element codes have difficulty in incorporating disc gyroscopic effects.

The transfer matrix method traditionally has suffered from numerical difficulties caused by ill conditioned matrices and singular branch roots. The authors have resolved many of these difficulties by proper scaling of the off diagonal elements of the transfer matrices, and incorporating a modified characteristic determinant to eliminate the singular solutions caused by branches. These changes have made it a more accurate and reliable method. By programming the modified transfer matrix method in the HP BASIC language on a microcomputer workstation, various numerical difficulties could be tracked and resolved by proper scaling techniques.

For example, in order to improve the accuracy of the transfer matrix method, many authors have used continuum beam theory. The transfer matrix elements then are represented by hyperbolic functions. The continuum formulation of the transfer matrix method is a numerical disaster even for large mainframe computers computing in double precision. With as few as 40 elements for a single span rotor, numerical difficulties have been obtained on main frame computers by this procedure. Therefore, the use of conventional continuum theory for the transfer matrix elements of a multi-spool rotor is not practical and should be avoided.

When one examines the theory of finite elements, the reduction of a continuum to a discrete mass model, and consistent mass matrix theory, one finds that a relatively few number of elements are required to represent a rotor span accurately. By proper mass lumping, for example, a uniform beam may be calculated within 1% accuracy for the *nth* mode by as few as 2n+1 stations. This is in contrast to the work of other researchers who have indicated that many more stations are required to accurately solve for the *nth* frequency. The difficulty here lies in the improper lumping of the mass stations.

Because of the ability to lump the rotor mass stations, many researchers such as Prohl and Lund have reduced the continuum transfer matrix procedure to a combination of a point matrix and a massless field matrix. The use of a massless field matrix results in a numerical method which is much better behaved than continuum theory. However, even with the massless field transfer matrix formulation, numerical difficulties will still be encountered with large rotor models over 100 sections for multi-span rotors. Although the hyperbolic functions have been eliminated, the unscaled transfer matrices may contain elements whose off diagonal terms are small in comparison to unity. This leads to computer round off errors and effects the rotor mode shapes at the far stations.

In this multi-spool program, flexible disks or branches may be incorporated on each rotor main span. In the application of the transfer matrix method, branches have traditionally caused numerical difficulties in critical speed calculations. The difficulties arise when a constrained branch mode falls within the operating speed of the rotor system. This causes a singularity in the characteristic polynomial and does not represent a true root of the system. The polynomial becomes infinite on either side of the singular point because of the local resonance of the branch. (A similar problem occurs with torsional vibration analysis of branched systems when using the standard transfer matrix method.) The singular behavior of the system characteristic determinant has been eliminated by multiplying the system characteristic matrix by the local characteristic matrices based on the constrained branch natural frequencies.

2. MATRIX TRANSFER METHOD

2.1 Transfer Matrix for a Single Station

Consider the discrete mass model of a single rotor. The lumped parameter model is obtained by dividing the rotor into a number of stations. The shaft segments are considered to be massless and weights are located at the point mass stations only.

An individual rotor section is composed of two types of elements represented by matrices. One is a lumped point-mass matrix which contains the gyroscopics and bearing reactions, and the other is a massless beam or field transfer matrix. The combination of the point-mass and beam field forms the complete section transfer matrix as shown in Fig. 1.

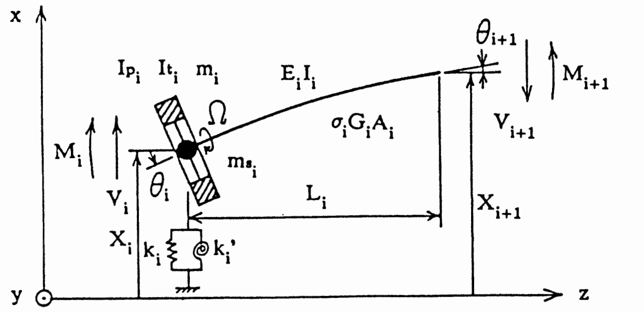


Fig. 1 Combination of Lumped Point-Mass and Beam Elements

The lumped point-mass has a shaft mass ms_i, a disk mass m_i, a polar moment of inertia Ip_i, and a transverse moment of inertia It_i. This point may have an elastic connection to the ground which consists of a linear spring constant of k_i and a rotational spring constant of k_i'.

The massless shaft section (length L_i) has a bending elasticity E_iI_i (E_i: Young's modulus, I_i: moment of inertia of area) and shearing elasticity $\sigma_iG_iA_i$ (σ_i : form factor, G_i: shearing modulus, A_i: cross sectional area). Where σ_i is given by,

$$\sigma_i = \frac{6(1+\nu)(1+r^2)^2}{(7+6\nu)(1+r^2)^2+(20+12\nu)r^2} \qquad (1)$$

where $r = d_i/D_i$ (D_i: outside diameter, d_i: inside diameter, ν: Poisson's ratio)

Let $\{Z\}_i=\{X_i,\theta_i,M_i,V_i\}^T$ be a left side state vector of *ith* station, where X_i is a displacement, θ_i is a slope, M_i is a moment and V_i is a shearing force acting at the *ith* station.

The transfer matrix $[Te]_i$ of this combined element is,

$$\{Z\}_i = [Fe]_i\,[Pe]_i\,\{Z\}_i^L = [Te]_i\,\{Z\}_{i-1} \qquad (2)$$

or

$$\begin{Bmatrix} X \\ \theta \\ M \\ V \end{Bmatrix}_i = \begin{bmatrix} 1+a_2a_5 & L+a_1a_4 & a_1 & a_2 \\ a_1a_5 & 1+a_3a_4 & a_3 & a_1 \\ a_5L & a_4 & 1 & L \\ a_5 & 0 & 0 & 1 \end{bmatrix}_i \begin{Bmatrix} X \\ \theta \\ M \\ V \end{Bmatrix}_{i-1} \qquad (3)$$

where

$$a_1 = L_i/(2E_iI_i), \qquad a_2 = L_i^3/(6E_iI_i) - L_i/(\sigma_iG_iA_i),$$

$$a_3 = L_i/(E_iI_i), \qquad a_4 = Ip_i\Omega\omega - It_i\omega^2 + k_i',$$

$$a_5 = m_i\omega^2 + ms_i\omega^2 - k_i \qquad (4)$$

and Ω is a spin speed, ω is a frequency of bending vibration.

The strain energy due to the shaft bending deformation Ub_i, shearing deformation Us_i and the spring deformation Uk_i are given by,

$$Ub_i = \frac{1}{2}\int_0^{L_i}\frac{M(z)^2}{E_iI_i}dz = \frac{1}{2E_iI_i}\left(M_L^2L + V_LM_LL^2 + \frac{1}{3}V_L^2L^3\right)_i \qquad (5)$$

$$Us_i = \frac{1}{2}\int_0^{L_i}\frac{V(z)^2}{\sigma_iG_iA_i}dz = \frac{V_i^2L_i}{2\sigma_iG_iA_i} \qquad (6)$$

$$Uk_i = \frac{1}{2}k_i X_i^2 + \frac{1}{2}k_i'\theta_i^2 \qquad (7)$$

The kinetic energy due to mass Tm_i and moment of inertia Tg_i are

$$Tm_i = \frac{1}{2}(m_i + ms_i)\omega^2X_i^2 \qquad (8)$$

$$Tg_i = \frac{1}{2}(It_i\omega - Ip_i\Omega)\omega\theta_i^2 \qquad (9)$$

where X_i and θ_i are the displacement and slope at the left side of mass point i, M_{i+1} and V_{i+1} are moment and shearing force at left side of mass point i+1.

2.2 Scaling the Matrix

In order to produce accurate eigenvalues and mode shapes by the matrix transfer method, scaling must be incorporated into the transfer matrix equations.

Essentially the field transfer matrix is composed of unity elements along the diagonal and the off diagonal terms are of the order L^n/EI; n=1,2,3. The off diagonal terms for models with a large number of rotor stations may become extremely small. When the large number of stations are multiplied together, numerical underflow may result in the off diagonal matrix terms. The numerical scaling procedure, introduced in this situation, transforms the off diagonal terms to quantities near unity.

Let the terms E_0, I_0 and L_0 represent the average rotor station values for Young's modulus, moment of inertia of area and station length respectively. The three terms are nondimensionalized and are used as follows:

$$\overline{X}_i = X_i/L_0, \qquad \overline{\Theta}_i = \theta_i, \qquad \overline{M}_i = \{L_0/(E_0I_0)\}M_i$$

$$\overline{V}_i = \{L_0^2/(E_0I_0)\}V_i, \qquad \overline{L}_i = L_i/L_0, \qquad \overline{E_iI_i} = E_iI_i/(E_0I_0),$$

$$\overline{G_iA_i} = \{L_0^2/(E_0I_0)\}G_iA_i, \qquad \overline{k}_i = \{L_0^3/(E_0I_0)\}k_i$$

$$\overline{k_i'} = \{L_0/(E_0I_0)\}k_i', \qquad \overline{m_i\omega^2} = \{L_0^3/(E_0I_0)\}m_i\omega^2,$$

$$\overline{ms_i\omega^2} = (\{L_0^3/(E_0I_0)\}ms_i\omega^2, \tag{10}$$

$$\overline{Ip_i\Omega\omega} = \{L_0/(E_0I_0)\}Ip_i\Omega\omega, \qquad \overline{It_i\omega^2} = \{L_0/(E_0I_0)\}It_i\omega^2 .$$

Eq.3 can be rewritten in a dimensionless form,

$$\{Z\}_i = [T]_i \{Z\}_{i-1}$$

$$\begin{Bmatrix} \overline{X} \\ \overline{\Theta} \\ \overline{M} \\ \overline{V} \end{Bmatrix}_i = \begin{bmatrix} 1+\overline{a_2a_5} & \overline{L+a_1a_4} & \overline{a_1} & \overline{a_2} \\ \overline{a_1a_5} & 1+\overline{a_3a_4} & \overline{a_3} & \overline{a_1} \\ \overline{a_5L} & \overline{a_4} & 1 & \overline{L} \\ \overline{a_5} & 0 & 0 & 1 \end{bmatrix}_i \begin{Bmatrix} \overline{X} \\ \overline{\Theta} \\ \overline{M} \\ \overline{V} \end{Bmatrix}_{i-1} \tag{11}$$

Where the scaled values of a_i are as follows:

$$\overline{a_1} = \overline{L_i}/(2\overline{E_iI_i}), \quad \overline{a_2} = \overline{L_i}^3/(6\overline{E_iI_i}) - \overline{L_i}/(\sigma_i\overline{G_iA_i}),$$

$$\overline{a_3} = \overline{L_i}/(\overline{E_iI_i}), \quad \overline{a_4} = \overline{Ip_i\Omega\omega} - \overline{It_i\omega^2} + \overline{k_i'},$$

$$\overline{a_5} = \overline{m_i\omega^2} + \overline{ms_i\omega^2} - \overline{k_i} \tag{12}$$

Eq.3 and Eq.11 are of the same form, dimensioned and dimensionless valuables need not be distinguished.

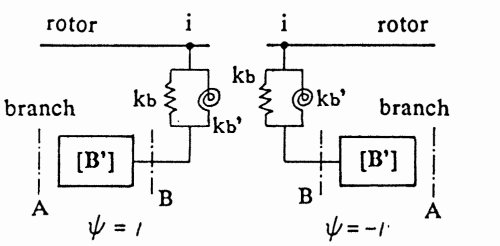
On determination of the average value of I_o, the following procedure is recommended for the hollow shaft.

$$I_0 = \frac{\pi}{64} \left\{ \left(\frac{1}{N} \sum_{i=1}^{N} D_i\right)^4 - \left(\frac{1}{N} \sum_{i=1}^{N} d_i\right)^4 \right\} \tag{13}$$

Where D_i and d_i are outside and inside diameter of the *ith* element, and N is the number of elements.

2.3 Branch Equations

If the rotor has a branch as shown in Fig.2, the contribution of the branch for the connection point of the rotor is considered as an elastic connection to the ground.



(a) Left Side Branch (b) Right Side Branch

Fig.2 Branch Model

The state vector of point A, $\{Z'\}_A$, represents the free end of the branch. The state vector of point B, $\{Z'\}_B$, represents the end of the branch connected to the rotor. The (') means that the state vector is described by the branch co-ordinates. $\{Z\}_i$ represents the left side state vector of point i on the rotor. $[T_B']$ is the transfer matrix of the branch; k_b is a linear spring element connecting the branch to the rotor; and k_b' is a rotational spring element at the connecting point of the branch to the rotor. The value of k_b=0 and/or k_b'=0 represents the program default value for a rigid connection implying that $k_b=\infty$ and/or $k_b'=\infty$. The sign of ψ is an indicator of the branch direction (ψ =1 for a left sided branch, and ψ =-1 for a right sided branch) as shown in Fig.2. The branch transfer matrix can be written as,

$$\{Z\}_i = [G]\,[C']_i\,[B']_i\,\{Z'\}_A \tag{14}$$

where [G] is co-ordinate transformation matrix, [C'] is a connection matrix of the branch to the rotor, as given by Eq. 15 and Eq. 16.

$$[G] = \begin{bmatrix} \psi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \psi \end{bmatrix} \tag{15}$$

$$[C']_i = \begin{bmatrix} 1 & 0 & 0 & -1/k_b \\ 0 & 1 & 1/k_b' & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

Let matrix [A] be defined as;

$$[A] = [G][C']_i[B']_i = \begin{bmatrix} [A11] & [A12] \\ [A21] & [A22] \end{bmatrix} \tag{17}$$

where

$$[A11] = \begin{bmatrix} (b_{11} - b_{41}/k_b)\psi & (b_{12} - b_{42}/k_b)\psi \\ b_{21} + b_{31}/k_b' & b_{22} + b_{32}/k_b' \end{bmatrix}$$

$$[A12] = \begin{bmatrix} (b_{13} - b_{43}/k_b)\psi & (b_{14} - b_{44}/k_b)\psi \\ b_{23} + b_{33}/k_b' & b_{24} + b_{34}/k_b' \end{bmatrix}$$

$$[A21] = \begin{bmatrix} b_{31} & b_{32} \\ b_{41}\psi & b_{42}\psi \end{bmatrix} \tag{18}$$

$$[A22] = \begin{bmatrix} b_{33} & b_{34} \\ b_{43}\psi & b_{44}\psi \end{bmatrix}$$

and b_{ij} (i,j=1,2,3,4) are the elements of matrix [B']. Because only moment and shear force are transferred from the branch to the connecting point, the transfer matrix at point i on the rotor is written as follows:

$$[Tb]_i = \begin{bmatrix} [U] & [0] \\ [K] & [U] \end{bmatrix}_i \tag{19}$$

where [U] is a unity matrix, [0] is a zero matrix. Considering the boundary conditions of the branch end, and comparing Eq.14 and Eq.19, we obtain the free branch end boundary conditions as follows,

$$[K] = [A21][A11]^{-1} \tag{20}$$

It is important to note that the determinant of the matrix [A11] gives us the constrained frequency determinant of the branch if the branch boundary conditions are assumed to be free at point A and fixed at the connection side to the rotor. This means that the det[A11] will be zero if the search frequency ω is equal to a constrained branch natural frequency under these boundary conditions. For example, consider the two degree of freedom system shown in Fig.3. This system is composed of a single mass (m) and spring (k) system with a connecting branch (mass m_B and spring k_B). The transfer matrix of this system using the branch formulation is given as follows:

$$\begin{Bmatrix} X \\ V \end{Bmatrix}_B = \begin{bmatrix} 1 & 0 \\ m\omega^2-k & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ m_B\omega^2/(1-m_B\omega^2/k_B) & 1 \end{bmatrix} \begin{Bmatrix} X \\ V \end{Bmatrix}_A \qquad (21)$$

$$= \begin{bmatrix} 1 & 0 \\ m\omega^2-k+(m_B\omega^2)/(1-m_B\omega^2/k_B) & 1 \end{bmatrix} \begin{Bmatrix} X \\ V \end{Bmatrix}_A$$

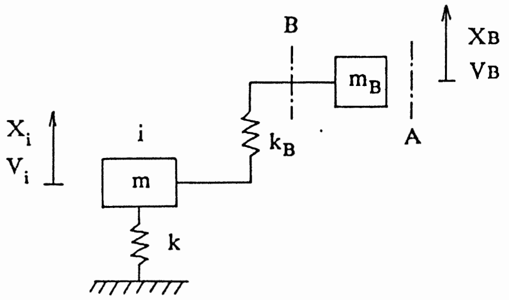


Fig.3 Two Degrees of Freedom Model

The natural frequency of the system is obtained by setting the function $\Delta(\omega)$ to zero.

$$\Delta(\omega) = (m\omega^2-k) + m_B\omega^2/(1-m_B\omega^2/k_B) \qquad (22)$$

However, the characteristic polynomial $\Delta(\omega)$ has a singular point when the frequency ω is equal to the branch constrained natural frequency $(k_B/m_B)^{\frac{1}{2}}$. This singular point does not represent a natural frequency of the two degree of freedom system, but corresponds to the constrained local resonance frequency of the attached branch.

Obviously the denominator of the second term of Eq. 22 is equivalent to the det[A11] of Eq. 20. For the case of multi-degree of freedom systems, branches may cause numerical difficulties when computing rotor critical speeds if the branch frequency is within the critical speed search vicinity. To avoid this problem, a modified characteristic function $\Delta'(\omega)$ should be introduced.

$$\begin{aligned} \Delta'(\omega) &= (1-m_B\omega^2/k_B)\Delta(\omega) \\ &= (m\omega^2-k)(1-m_B\omega^2/k_B)+m_B\omega^2 \end{aligned} \qquad (23)$$

Fig. 4 shows the comparison of the roots of $\Delta(\omega)$ and $\Delta'(\omega)$ as a function of ω. The points where $\Delta(\omega)=0$ represent system natural frequencies. $\Delta(\omega)$ has a singular point at $\omega_B = (k_B/m_B)^{\frac{1}{2}}$, which is the natural frequency of the branch, while $\Delta'(\omega)$ has no such point.

In general, when calculating the value of the frequency determinant, the magnitude of the frequency determinant is not of concern. The objective is to find the roots or frequencies which make the determinant equal to zero. These zeros of the characteristic determinant correspond to the system critical speeds. To avoid numerical difficulties caused by branches, the matrix [K] of Eq. 20 should be modified to [K'] as follows, using the definition of a matrix inverse.

$$[K'] = \det[A11]\,[K] = \det[A11]\,([A21][A11]^{-1})$$

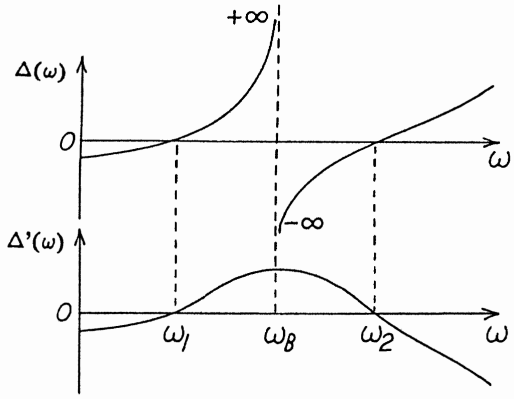


Fig.4 Comparison of $\Delta(\omega)$ and $\Delta'(\omega)$

Note that when ω corresponds to a constrained branch natural frequency det[A11] is zero, while $[A11]^{-1}$ is singular. This numerical difficulty is removed by using the definition of the inverse of a matrix that $[A11]^{-1} = \mathrm{Adj}[A11]/\det[A11]$. This results in a modified value of [K'] as follows;

$$[K'] = [A21]\,\mathrm{Adj}[A11] \qquad (24)$$

In this formulation, [K'] has the singular points removed. Also branch transfer matrix Eq. 19 should be modified to be:

$$[Tb']_i = \det[A11]\,[Tb]_i = \begin{bmatrix} \det[A11][U] & [0] \\ [K'] & \det[A11][U] \end{bmatrix} \qquad (25)$$

Note that the branch contribution to the rotor transfer matrix requires the use of Eq. 25 to determine the critical speeds, and Eq. 19 to calculate the mode shapes and energy balance.

The combined transfer matrix $[Tc]_i$ as shown in Fig. 5, is given by,

$$[Tc]_i=[Te]_i[Tb]_i \qquad (26)$$

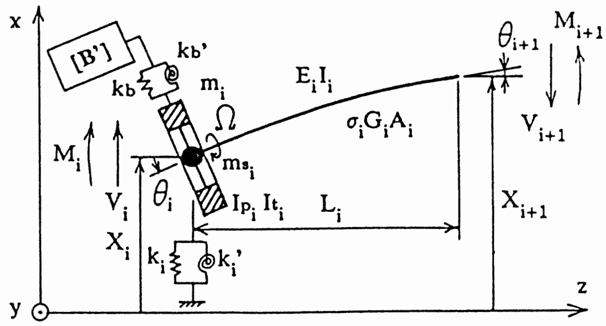


Fig.5 Combined Element with Branch

In the procedure for critical speed and mode shape, the determination branch transfer matrix $[Tb]_i$ should be replaced by $[Tb']_i$.

2.5 Transfer Matrix for Multi-Rotor System

Consider the discrete mass model of a multi-spool rotor system as shown in Fig. 6. This system is composed of three rotors with coincident and parallel axes. This model is constructed such that the third rotor, which is the casing, has elastic connections to ground, and the other shafts are interconnected by internal elastic bearings. The former bearings are called "external bearings" and the latter are named internal or "differential bearings".

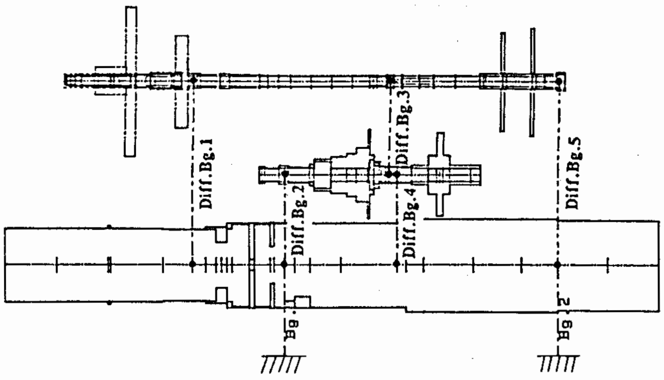


Fig.6 Three-Span-Rotor Model

Assuming undamped free vibrations, the dynamic behavior of each rotor is represented by a 4x4 alley transfer matrix. The complete three-spool rotor system shown in Fig. 6 is represented by a 12x12 system transfer matrix. In the case of forced response with damping, the alley size would increase to 2n+1x2n+1 complex matrices.

To analyze the multi-rotor system functionally, Fig. 6 should be modified as shown in Fig. 7.

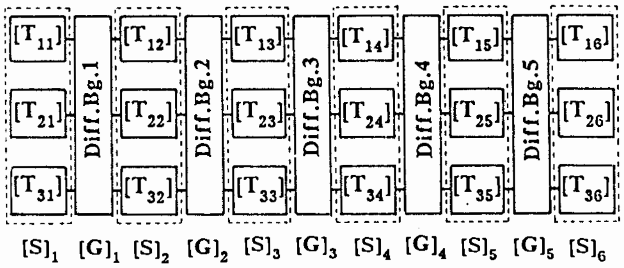


Fig.7 Analytical Model of the Three-Span-rotor model

In this multi-spool rotor analysis, the most important information is the location of differential bearings, and that each rotor should be divided into a number of sections at the differential bearing locations. (Each section may contain a number of single rotor stations.) For example, rotor No. 1 of Fig. 6, which has three differential bearings, should be divided into four sections. The Three-spool rotor system of Fig. 6 has five differential bearings, but none of these are connected to the same rotor or the same station. These rotors, therefore, should be divided into seven sections as shown in Fig. 7.

However, rotor No. 1 is divided into four sections, and the remaining two sections should be dummy sections. (The transfer matrix of the dummy section is the unity matrix.) Fig. 7 shows the modified three spool rotor model, where [Tij] means jth section transfer matrix of rotor No. i. In this case [T13], [T15], [T22], [T26] and [T34] are dummy sections. The file group of transfer matrices [T1j], [T2j] and [T3j] make a space matrix, while the differential bearings make a grid matrix for the multi-rotor system.

Generally, for the n-spool rotor system in which each grid may have up to n(n-1)/2 differential bearings, the space matrix $[S]_i$ consists of each rotor's transfer matrices of the section, and the grid matrix $[G]_i$ consists of number of linear or rotational springs at the grid, can be written as;

$$[S]_i = \begin{bmatrix} [T1i] & 0 & 0 & \cdots & 0 \\ 0 & [T2i] & 0 & \cdots & 0 \\ 0 & 0 & [T3i] & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & [Tni] \end{bmatrix} \tag{27}$$

$$[G]_i = \begin{bmatrix} [G11] & [G12] & [G13] & \cdots & [G1n] \\ [G21] & [G22] & \cdots & \cdots & \cdots \\ [G31] & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ [Gn1] & \cdots & \cdots & \cdots & [Gnn] \end{bmatrix}_i \tag{28}$$

where

$$[Gij] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -k_{ij}' & 0 & 0 \\ k_{ij} & 0 & 0 & 0 \end{bmatrix} \quad (i \neq j)$$

$$[Gii] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b_i' & 1 & 0 \\ -b_i & 0 & 0 & 1 \end{bmatrix} \quad (i = i) \tag{29}$$

k_{ij} : linear differential spring constant (connects rotor No. i and No. j)
k_{ij}' : rotational differential spring constant (connects rotor No. i and No. j)
b_i : sum of linear differential spring constants which connects rotor No. i to other rotors
b_i' : sum of rotational differential spring constants which connects rotor No. i to other rotors

The transfer matrix of this multi-spool rotor system is given by multiplying the space matrices and grid matrices as follows;

$$[T] = [S]_{M+1}[G]_M[S]_M \ldots [G]_2[S]_2[G]_1[S]_1 \tag{30}$$

where M is the number of grids in the multi-spool-rotor system. The alley size of matrix [T] should be 4nx4n (n: number of rotors).

Assuming that the boundary conditions of this system are that each rotor has free ends, the frequency determinant of the transfer matrix of Eq. 30 is written as,

$$\Delta(\omega) = \begin{vmatrix} t_{3,1} & t_{3,2} & \cdots & t_{3,4n-3} & t_{3,4n-2} \\ t_{4,1} & t_{4,2} & \cdots & t_{4,4n-3} & t_{4,4n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{4n-1,1} & t_{4n-1,2} & \cdots & t_{4n-1,4n-3} & t_{4n-1,4n-2} \\ t_{4n,1} & t_{4n,2} & \cdots & t_{4n,4n-3} & t_{4n,4n-2} \end{vmatrix} \tag{31}$$

where t_{ij} means the element of matrix [T].

To avoid numerical difficulties with rotors containing branches, the transfer matrix [Tij] of each rotor section should be replaced by modified matrix using Eq. 24 and Eq. 25 when the frequency determinant is calculated.

To determine the state vectors, the left end displacement of rotor No. 1 is set to one. The other elements of the left end state vector are obtained by,

$$\begin{matrix} \theta_1 \\ X_2 \\ \theta_2 \\ \cdots \\ X_n \\ \theta_n \end{matrix} \begin{bmatrix} t_{4,2} & t_{4,5} & \cdots & t_{4,4n-3} & t_{4,4n-2} \\ t_{7,2} & t_{7,5} & \cdots & t_{7,4n-3} & t_{7,4n-2} \\ t_{8,2} & t_{8,5} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{4n-1,2} & t_{4n-1,5} & \cdots & t_{4n-1,4n-3} & t_{4n-1,4n-2} \\ t_{4n,2} & t_{4n,3} & \cdots & t_{4n,4n-3} & t_{4n,4n-2} \end{bmatrix} \begin{matrix} t_{4,1} \\ t_{7,1} \\ \cdots \\ \cdots \\ t_{4n-1,1} \\ t_{4n,1} \end{matrix} \tag{32}$$

3. OUTLINE OF THE ALGORITHM

The computer algorithm used to calculate undamped critical speeds of multi-spool rotors with multiple branches consists of the following 16 steps:

step 1: **Read data** of multi-spool rotors
- (1) number of rotors and branches
- (2) each rotor and branch shape and material data
- (3) whirl condition (forward and backward & speed)
- (4) number of differential bearing grids
- (5) differential bearing data
- (6) speed range and number of modes

step 2: **Printout** all data and **graphic display** of rotor shapes

step 3: Determine **combined element matrix** $[Te]_{ij}$
($i=1, 2,...N_j$, Nj: number of stations of rotor j
$j=1, 2....N$, N: number of rotors)

step 4: Determine **branch matrix** $[B]_{ij}$
($i=1, 2,...N_{Bj}$, NBj: number of stations of branch j
$j=1, 2,...N_B$, NB: number of branches)

step 5: Determine **section matrix** $[T_S]_{ij}$
($i=1, 2,...M+1$, M: number of grids, $j=1, 2,...N$)

step 6: Determine **space matrix** $[S]_i$
($i=1, 2,...M+1$)

step 7: Determine **grid matrix** $[G]_i$
($i=1, 2,...M$)

step 8: Determine **system transfer matrix**
[T] and [T']

step 9: Calculate **frequency determinant** of
[T'] and determine **critical speeds**

step 10: Determine **state vector** $\{Z_S\}^L{}_1$ of left end of the multi-spool rotors

step 11: Determine **state vector** $\{Z_S\}^L{}_i$
($i=1, 2,...M+1$)

step 12: Determine **state vector** $\{Z\}_{ij}$
($i=1, 2,...M$, $j=1, 2,...N$)

step 13: Determine **branch state vector** $\{Z_B\}_{ij}$
($i=1, 2,...N_{Bj}$, $j=1, 2,...N_B$)

step 14: Calculate maximum **strain energies** U_{si}, U_{bi} and U_{ki}; and maximum **kinetic energies** T_{mi} and T_{gi} for each station of each rotor and each branch

step 15: Calculate **total strain energy** U_{max} and **total kinetic energy** T_{max}, then check the accuracy ($U_{max}=T_{max}$)

step 16: **Printout** state vector and energy distribution, and **graphic display** of mode shapes

4. CONCLUSIONS

The theory and computer algorithm to accurately calculate the undamped critical speeds of multiple-span rotors with multiple flexible branches is represented. The transfer matrix method for the critical speed calculation is a straight forward procedure. However, numerical difficulties are often encountered in practice due to round-off errors and attached branches.

The rotor transfer matrix formulation may be based on continuum theory or on a discrete lumped mass model. The continuum formulation of a distributed beam mass element leads to transcendental functions. Any attempt to solve for the critical speeds (or unbalance response, stability) of a large multi-span rotor system using transcendental transfer matrices leads to numerical round-off error truncation problems even for modest sized systems. The lumped mass model formulation will permit larger models, but it too suffers from numerical round off in multi-span models.

By introducing dimensionless values of slope, moment and shear, the authors have incorporated scaling factors into the transfer matrices which normalize the off diagonal terms. This allows accurate calculations of 5 level rotor systems on engineering work stations carrying 15 digits of precision.

Another numerical difficulty of the transfer matrix method is the generation of singular roots caused by branch sections. The authors demonstrate how this problem can be resolved by redefining the characteristic determinant.

REFERENCES

Eshleman, B.L., "Flexible Rotor-Bearing System Dynamics I. Critical Speeds and Response of Flexible Rotor Systems," *ASME*, 1972.

Gunter, E.J., Li, D.F., and Barrett, L.E., "Unbalance Response of a Two Spool Gas Turbine Engine with Squeeze Film Bearings," *ASME Gas Turbine Conference*, March 1981, 81-GT-219.

Kawamo, K., Malsuhura, Y., and Inoue, T., "Analysis of Lateral Vibration Characteristics of Rotating Shafts with Flexible and Axi-Asymmetric Bearings," *1975 Joint JSME-ASME Mechanics Western Conference.*

Kikuchi, K., "Analysis of Unbalance Vibration of Rotating Shaft with Many Bearings and Disks," *JSME Bulletin*, June 16, 1969.

Koenig, E.C., "Analysis for Calculating Lateral Vibration Characteristics of Rotating Systems with any Number of Flexible Supports, Part 1 - The Method of Analysis," *Journal of Appl. Math.*, Dec. 1961, Vol. 28, pp. 585-590.

Li, D.F., and Gunter, E.J., *Undamped Critical Speed Analysis of Dual-Level Rotors Including Lateral, Axial and Torsional Vibration Coordinates*, School of Engineering and Applied Science, Univ. of Virginia, Dept. of Mech. and Aerospace Engineering, Report No. UVA 1528140/MAE 78/114, Sept. 1978.

Lund, J.W. and Orcutt, F.K., "Calculations and Experiments on the Unbalance Response of a Flexible Rotor," *Journal of Engineering for Industry*, Nov. 1967.

Miyachi,T., "A Parametric Study of the Unbalance Vibration of the Rotor Supported in a Light, Flexible Casing." *1983 Tokyo International Gas Turbine Congress*, 83-Tokyo-IGTC-125, pp. 969.

Myklestad, N.O., "A New Method of Calculating Natural Modes of Uncoupled Bending Vibration of Airplane Wings and Other Beams," *Journal of the Aeronautical Sciences*, pp. 153-162, April 1944.

Pestel, E. and Leckie, F., *Matrix Methods in Elastomechanics*, McGraw-Hill, 1963.

Prohl, M.A., "A General Method for Calculating Critical Speeds of Flexible Rotors," *Journal of Applied Mechanics*, Trans. ASME, Vol. 67, 1945, pp. A142-A148.

Ruhl, B., *Dynamics of Distributed Parameter Rotor Systems: Transfer Matrix and Finite Element Techniques*, Ph.D. Thesis, Cornell University, 1970.

Salamone, D.J., and Gunter, E.J., "Effects of Shaft Warp and Disk Skew on the Synchronous Unbalance Response of a Multi-Mass Rotor in Fluid Film Bearings," *ASME Topics in Fluid Film and Rotor Bearing System Design and Optimization*, pp. 79-107, 1978.

Thompson, W., *Vibration Theory with Applications*, Prentice-Hall, 1972.