DYNAMIC STABILITY OF ROTOR-BEARING SYSTEMS

Edgar J. Gunter, Jr.

Department of Mechanical Engineering
University of Virginia

Prepared under contract
for Lewis Research Center
Cleveland, Ohio
by The Franklin Institute Research Laboratories
Philadelphia, Pennsylvania

Scientific and Technical Information Division
OFFICE OF TECHNOLOGY UTILIZATION 1966
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Washington, D.C.

Appendix D

Hydrodynamic Bearing Equations

D.1 DERIVATION OF THE GENERAL REYNOLDS EQUATION FOR AN ISOTHERMAL COMPRESSIBLE FLUID

Consider the forces acting on a small volume element τ. The equations of motion of the volume will be

$$\iiint_V \rho \frac{D\vec{u}}{Dt} d\tau = \iint_S \overset{\nu_i}{\vec{T}} \cdot \vec{n}\, ds + \iiint_V \vec{F}\, d\tau \qquad \text{(D.1)}$$

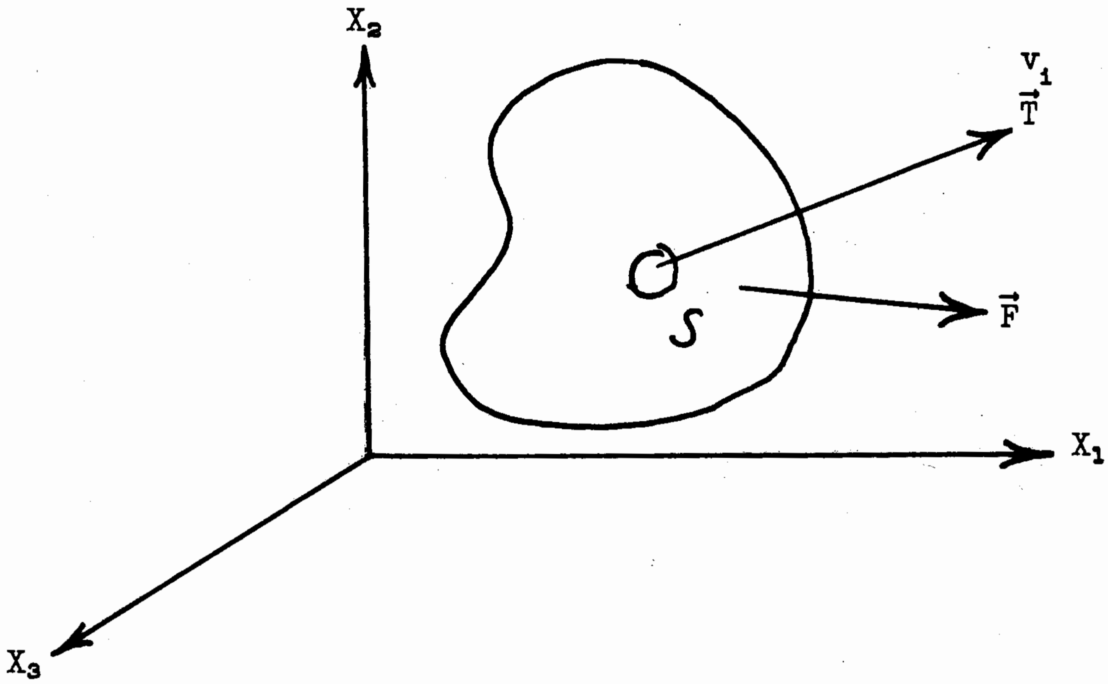

FIGURE D.1.—Forces acting on a small volume element.

Where

$\vec{F}$ = body force vector

$\vec{T}$ = traction vector $= \sigma_{ij} \nu_j \vec{n}_i$

By employing Gauss's theorem Eq. (D.1) becomes

$$\iiint_V \left[\rho \frac{Du_i}{Dt} - \frac{\partial \sigma_{ij}}{\partial X_j} - F_i\right] d\tau = 0 \tag{D.2}$$

Since the volume of integration is arbitrary, then

$$\rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial X_j} + F_i; \qquad i = 1, 2, 3 \tag{D.3}$$

Let

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij} \tag{D.4}$$

where τ_{ij} = viscous shear stresses.

If the fluid is assumed to be Newtonian, then the viscous shear stresses are linearly related to the rate of shear strain. This is represented by

$$\tau_{ij} = C_{ijkl}\epsilon_{kl} \tag{D.5}$$

If the fluid is also isotropic, then the fourth-order tensor C_{ijkl} is symmetric and invariant under coordinate transformation and is given by

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] + \gamma[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}] \tag{D.6}$$

Hence the stress-strain rate for an isotropic Newtonian fluid is given by

$$\tau_{ij} = \lambda\delta_{ij}\Delta + 2\mu\epsilon_{ij} \tag{D.7}$$

Where

λ, μ = Lamé constants

ϵ_{ij} = symmetric strain rate tensor

$= \frac{1}{2}[u_{i,j} + u_{j,i}]$

Δ = dilatation $= \epsilon_{ii} = u_{i,i}$

Contraction of Eq. (D.7) results in

$$\tau_{ii} = [3\lambda + 2\mu]\Delta \tag{D.8}$$

In the case of an incompressible fluid where the dilatation Δ is zero, then the sum of the viscous normal stresses τ_{ii} is zero. If we assume that τ_{ii} will be zero even for a compressible mediam, then

$$3\lambda + 2\mu = 0 \tag{D.9}$$

$$\lambda = -2/3\mu$$

This is known as Stokes approximation and eliminates one of the Lamé constants from the governing equations of motion. This approximation has been shown to be true only for the case of a monatomic gas, but usually results in only higher order deviations for most gases at normal temperature and pressure. This assumption is invalid in regions where large pressure or velocity gradients exist. As an example, the assumption breaks down in the immediate vicinity of a supply orifice to an externally pressurized bearing if a shock wave occurs.

$$\tau_{ij} \underset{(D.7,\,9)}{=} \mu\left[-\frac{2}{3}\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i}\right] \tag{D.10}$$

Therefore the equations of motion are:

$$\rho\frac{Du_i}{Dt} \underset{(D.3,\,4,\,10)}{=} -\frac{\partial P}{\partial X_i} + F_i + \frac{\partial}{\partial X_j}\left[\mu\left(-\frac{2}{3}\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i}\right)\right] \tag{D.11}$$

If the viscosity μ is not a function of the coordinates X, then Eq. (D.11) reduces to

$$\frac{\rho Du_i}{Dt} = -\frac{\partial P}{\partial X_i} + F_i + \mu\left[\frac{1}{3}u_{j,ij} + u_{i,jj}\right] \tag{D.12}$$

If the body forces F_i are zero, then Eq. (D.12) relates the inertia forces to the rates of change of the hydrostatic pressure and viscous shear stresses. The major assumption in the formulation of a lubrication problem is that the flow is laminar. This is possible only if the inertia terms of the left-hand side of Eq. (D.12) are small in comparison to the viscous shear forces. This is equivalent to the statement that the Reynolds number is less than 1

$$R_e^* = \frac{\text{Inertia forces}}{\text{Viscous forces}} \approx \frac{\rho\frac{U^2}{L}}{\mu\frac{U}{h^2}} = \frac{UL}{\nu}\left(\frac{h}{L}\right)^2$$

Where

R_e^* = reduced Reynolds number
U = velocity
L = characteristic bearing length
h = characteristic film thickness
ν = kinematic viscosity

If

$$R_e^* \ll 1$$

then

$$\frac{\partial P}{\partial X_i} = \mu \left[\frac{1}{3} u_{j,ij} + u_{i,jj} \right] \tag{D.13}$$

Assign for X_i and u_i the following orders of magnitude:

$$X_1 = L \qquad u_1 = U$$

$$X_2 = h \qquad u_2 = \delta U$$

$$X_3 = L \qquad u_3 = U$$

where

$$\left(\frac{h}{L}\right) \quad \text{and} \quad \left(\frac{\delta U}{U}\right) \ll 1$$

As an example, let $i = 1$, in Eq. (D.12)

$$\frac{\partial P}{\partial X_1} = \mu \left[\frac{1}{3} \left(\frac{\partial^2 u_1}{\partial X_1 \partial X_1} + \frac{\partial^2 u_2}{\partial X_1 \partial X_2} + \frac{\partial^2 u_3}{\partial X_1 \partial X_3} \right) + \frac{\partial^2 u_1}{\partial X_1^2} + \frac{\partial^2 u_1}{\partial X_2^2} + \frac{\partial^2 u_1}{\partial X_3^2} \right] \tag{D.14}$$

It is seen that the term $\partial^2 u_1 / \partial X_2^2$ is an order of magnitude higher than the other terms. Hence, Eq. (D.13) reduces to:

$$\frac{\partial P}{\partial X_1} = \mu \frac{\partial^2 u_1}{\partial X_2^2} \tag{D.15}$$

Likewise for $i = 2$ and 3

$$\frac{\partial P}{\partial X_2} = 0 \tag{D.16}$$

$$\frac{\partial P}{\partial X_3} = \mu \frac{\partial^2 u_3}{\partial X_2^2} \tag{D.17}$$

For convenience let

$$X_1 = X; \qquad u_1 = u$$

$$X_2 = Y; \qquad u_2 = V$$

$$X_3 = Z; \qquad u_3 = W$$

Equations (D.14–D.16) may be written as

$$\frac{\partial P}{\partial X}=\mu\frac{\partial^2 u}{\partial Y^2} \tag{D.18}$$

$$\frac{\partial P}{\partial Y}=0 \tag{D.19}$$

$$\frac{\partial P}{\partial Z}=\mu\frac{\partial^2 W}{\partial Y^2} \tag{D.20}$$

Since $\partial P/\partial Y \underset{(D.18)}{=} 0$, the fluid film pressure may be considered as uniform across the film thickness. Equations (D.17) and (D.19) may be integrated directly and upon application of the following boundary conditions:

$$y=0 \qquad u(0)=U_1; \qquad W(0)=W_1$$

$$y=h \qquad u(h)=U_2; \qquad W(h)=W_2$$

(where $h=h(X,z,t)$ is the film thickness distribution)

$$u=\frac{1}{2\mu}\left(\frac{\partial P}{\partial X}\right)(Y-h)\,Y+U_1\left[\frac{h-Y}{h}\right]+U_2\frac{Y}{h} \tag{D.21}$$

$$W=\frac{1}{2\mu}\left(\frac{\partial P}{\partial Z}\right)[Y-h]\,Y+W_1\left[\frac{h-Y}{h}\right]+\frac{W_2 Y}{h} \tag{D.22}$$

Equations (D.20) and (D.21) are insufficient to formulate the lubrication problem. Another relationship is required. This is the continuity equation which is the statement of the conservation of mass in an elemental volume and is given by

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial X}+\frac{\partial(\rho V)}{\partial Y}+\frac{\partial(\rho W)}{\partial Z}=0 \tag{D.23}$$

If we assume a compressible isothermal fluid film which obeys the perfect gas laws, then

$$P=\mathscr{K}\rho$$

and Eq. (D.23) becomes:

$$\frac{\partial(PV)}{\partial Y}=-\left[\frac{\partial P}{\partial t}+\frac{\partial(Pu)}{\partial X}+\frac{\partial(PW)}{\partial Z}\right] \tag{D.24}$$

Integrate Eq. (D.23) across the fluid film

$$PV\Big|_{y=0}^{y=h} = -\int_0^{h(x,z,t)} \left[\frac{\partial P}{\partial t} + \frac{\partial (Pu)}{\partial X} + \frac{\partial (PW)}{\partial Z}\right] dy \tag{D.25}$$

In order to perform the above integration, it is necessary to place the derivatives with respect to X and Z outside the integration sign. To accomplish this we will use the Leibniz rule for differentiating under the integral sign when the limits of integration are a function of the current variable itself. If

$$I(\beta(x), \alpha(x), X) = \int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy$$

then

$$\frac{dI}{dX} = \frac{\partial I}{\partial X} + \frac{\partial I}{\partial \beta(x)}\left(\frac{\partial \beta}{\partial X}\right) + \frac{\partial I}{\partial \alpha}\left(\frac{\partial \alpha}{\partial X}\right)$$

Hence:

$$\frac{dI}{dX} = \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial X}\, dy + f\Big|_{y=\beta(x)} \frac{\partial \beta}{\partial X} - f\Big|_{y=\alpha(x)} \frac{\partial \alpha}{\partial X} \tag{D.26}$$

Thus:

$$\int_0^h \frac{\partial}{\partial X}(Pu)\,dy = \frac{\partial}{\partial X}\int_0^h Pu\,dy + Pu\Big|_{y=h} \frac{\partial h}{\partial X}$$

and

$$\int_0^h \frac{\partial}{\partial Z}(PW)\,dy = \frac{\partial}{\partial Z}\int_0^h PW\,dy + PW\Big|_{y=h} \frac{\partial h}{\partial Z} \tag{D.27}$$

After integrating Eq. (D.25) using the velocity profiles and rearranging

$$\frac{1}{6\mu}\left[\frac{\partial}{\partial X}\left(Ph^3\frac{\partial P}{\partial X}\right) + \frac{\partial}{\partial Z}\left(Ph^3\frac{\partial P}{\partial Z}\right)\right] = 2P(V_2 - V_1) + 2\frac{\partial P}{\partial t} h$$

$$+ h\left\{\frac{\partial}{\partial X}[P(U_2 + U_1)] + \frac{\partial}{\partial Z}[P(W_2 + W_1)]\right\} + P(U_1 - U_2)\frac{\partial h}{\partial X}$$

$$+ P(W_1 - W_2)\frac{\partial h}{\partial Z} \tag{D.28}$$

The above equation represents the general three-dimensional Reynolds equation as applied to an isothermal compressible fluid film.

D.2 DISCUSSION OF ASSUMPTIONS INVOLVED IN THE DERIVATION OF REYNOLDS EQUATION

The derivation of Reynolds equation as applied to a compressible isothermal fluid film proceeds from the consideration of the sum of the inertia, body forces, and tractions acting over a volume of fluid. By means of Gauss' theorem, the surface traction integral is converted to a volume integral to obtain the partial differential equations as applied to an infinitesimal element. The first assumption employed is:

(a) The fluid is Newtonian.

This implies that the fluid properties are invariant (isotropic) or unchanging in any direction and also that the stress tensor is linearly related to the strain rate tensor. (For fluids such as greases, the stress-strain rate is not linear at low shear rates and hence the media is non-Newtonian.)

By assuming a linear stress-strain, rate relationship, two Lamé constants are required to express the viscous shear stress in terms of shear strain rate (as an example, in solid mechanics these two constants are usually expressed in terms of E, Young's modulus, and ν, Poisson's ratio, or E and G = the shear modulus). The constants appearing in Eq. (D.7) are λ and μ the fluid viscosity. For the case of an incompressible fluid, the second Lamé constant λ does not enter into the equations, since the dilatation is zero (see Eq. (D.8)). In order to remove λ, it is assumed that—

(b) The bulk modulus $(3\lambda + 2\mu) = 0$.

By this means we are able to express λ in terms of μ. This is known as Stokes approximation, and thus the pressure P is independent of the dilatation. This assumption has shown to be valid only for a monatomic gas in which higher order molecular collisions are neglected and the gas is not an extreme pressure condition. In the normal gas bearing application, where the flow is laminar, the dilatation Δ is small and only secondary errors accrue. The assumption is invalid in regions where there are high velocity or pressure gradients. Such regions would be at the leading or trailing edge of a partial journal bearing where high-velocity gradients exist. Another region in which the assumption breaks down is in the immediate vicinity of a supply orifice for an externally pressurized bearing. In this case it is possible to have a local shock front formed downstream, and associated with it would be high-pressure gradients.

Equation (D.12) is usually referred to as the Navier-Stokes equations and represents three highly nonlinear partial differential equations. As such, no general solutions are available for Eq. (D.12) in its present form. To reduce the complexity of Eq. (D.12), the following assumptions are made:

(c) The effect of the body forces F_i are negligible.

For fluid film bearings with the body forces due to gravity only, this is true. In the case where the body forces are exerted by magnetic effects (magnetohydrodynamics), the forces F_i can be sizable. The key assumption in reducing the complexity of the general Navier-Stokes equations in the derivation Reynolds equation is the assumption that the flow is laminar viscous and the inertia effects are small in comparison to the viscous shear forces.

This assumption is equivalent to the statement that

(d) The reduced Reynolds number R_e^* is much less than unity.

This permits us to set the left-hand side of Eq. (D.12) equal to zero. To demonstrate the validity of statement (d), the reduced Reynolds number will be calculated for a typical gas bearing.

Example

The effective Reynolds number must be less than one for the Reynolds equation to be valid, or

$$R_e^* \ll 1$$

where

$$R_e^* = \frac{UL}{\nu}\left(\frac{h}{L}\right)^2$$

The effective Reynolds number will be calculated corresponding to typical operating conditions of a pivoted pad gas bearing experimental test rotor:

$N = 18\,000$ rpm
R = radius of rotor = 2 in.
$U = \frac{2\pi R}{60} N = 3768$ in./sec
$\mu = 2.61 \times 10^{-9}$ lb-sec/in.^{2}
$P = 14.7$ psia
$T = 70°$ F
$\rho = \frac{P}{gRT} = 1.1 \times 10^{-7}$ lb-sec^2/in.^{4}
$h = 0.001$ in. = average shoe film thickness
$\nu = \frac{\mu}{\rho} = 2.38 \times 10^{-2}$

$$L = R\alpha = 3.3 \text{ in.}$$

$$R_e^* = \frac{(3.768)(3.3)}{2.38 \times 10^{-2}} \left(\frac{1 \times 10^{-3}}{3.3}\right)^2 = 4.72 \times 10^{-2}$$

or

$$R_e^* \approx 0.05$$

Thus the assumption that the inertia terms are small in comparison to the viscous shear forces is valid and hence may be neglected in the range of operation considered. It has been pointed out by Constantinescu and Gross that in cases where R_e^* exceeds 1, the error induced by neglecting the contribution of the inertia terms is still small, acting so as to increase the bearing load capacity and friction losses.

D.3 DERIVATION OF FILM THICKNESS BETWEEN JOURNAL AND BEARING

Consider the triangle O_b, R, O_j of Fig. D.2

$$\cos \gamma = \frac{R - e_r}{R + C - h} \tag{D.29}$$

where

$e_r = e \cos(\theta - \phi - \gamma)$
h = bearing film thickness

If $C/R \ll 1$, then:

(a) $\cos \gamma \approx 1.0$
(b) $e_r \approx e \cos(\theta - \phi)$

Equation (D.29) becomes

$$1.0 = \frac{R - e \cos(\theta - \phi)}{R + C - h(\theta)} \tag{D.30}$$

Solving for $h(\theta)$

$$h(\theta) = C\,[1 + \epsilon \cos(\theta - \phi)] \tag{D.31}$$

where

ϵ = eccentricity ratio = e/C

Let the eccentricity vector $\vec{e}$ be given by

$$\vec{e} = e\vec{e}_r = -X\vec{n}_x + Y\vec{n}_y \tag{D.32}$$

Taking the dot product of Eq. (D.32) with respect to $\vec{n}_x$ and $\vec{n}_y$

$$e\vec{e}_r = (-X\vec{n}_x + Y\vec{n}_y) \cdot (\vec{n}_x; \vec{n}_y)$$

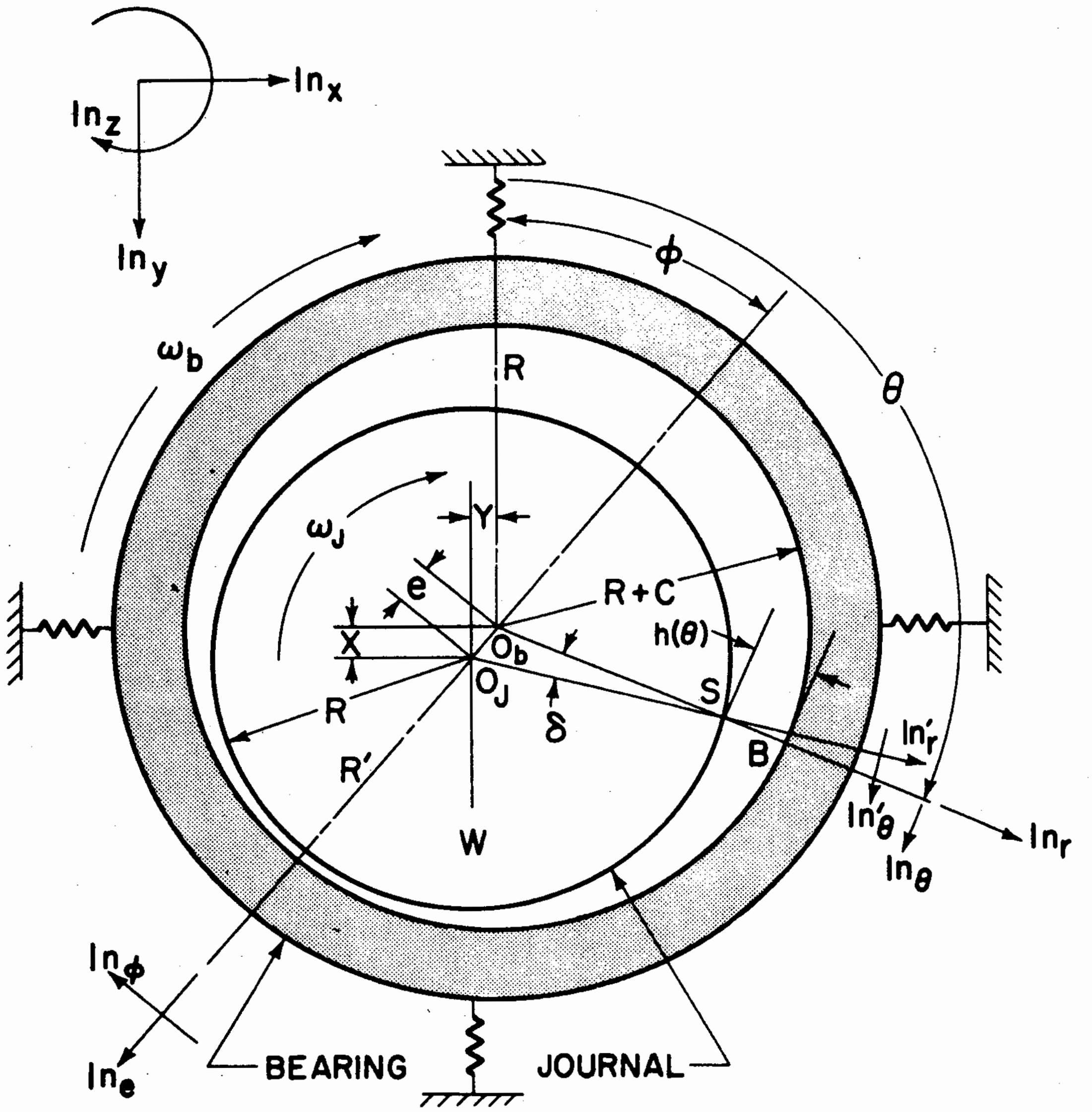

FIGURE D.2.—Bearing geometry.

yields

(a) $-e \sin \phi = X$

(b) $e \cos \phi = Y$ (D.33)

The film thickness in terms of Cartesian coordinates is given by

$$h(\theta) \underset{(D.31,\, D.33)}{=} C + Y \cos \theta + X \sin \theta$$

D.4 KINEMATICS

D.4.1 Journal Motion

The velocity of an arbitrary point S on the journal surface in reference frame R is given by

$$\overset{R\to S}{\mathbf{V}} = \overset{R\to O_j/O_B}{\mathbf{V}} + \overset{R\to S/O_j}{\mathbf{V}} \tag{D.34}$$

where

R = reference frame fixed in bearing

$\overset{R\to O_j/O_B}{\mathbf{V}}$ = velocity of journal center O_j relative to bearing O_B in R

$\overset{R\to S/O_j}{\mathbf{V}}$ = velocity of point S relative to the journal center O_j in R

$$\overset{R\to O/O_B}{\mathbf{V}} = \frac{^R d}{dt}[e\vec{e}_r] = \dot{e}\vec{e}_r + {}^R\vec{\omega}^{R'} \times (e\vec{e}_r)$$

where

${}^R\vec{\omega}^{R'}$ = angular velocity of rotating reference frame R' in R

$= \dot{\phi}\vec{n}_z$ = precession or whirl speed

$\overset{R\to O_j/O_B}{\mathbf{V}} = \dot{e}\vec{e}_r + e\dot{\phi}\vec{e}_\phi$

transforming to the $\vec{n}_r$, $\vec{n}_\theta$ vector set

$$\overset{R\to O_j/O_B}{\mathbf{V}} = -[\dot{e}\cos(\theta-\phi) + e\dot{\phi}\sin(\theta-\phi)]\,\vec{n}_r + [\dot{e}\sin(\theta-\phi) - e\dot{\phi}\cos(\theta-\phi)]\,\vec{n}_\theta \tag{D.35}$$

If the journal eccentricity is expressed in Cartesian coordinates, then the velocity of the journal center is

$$\overset{R\to O_j/O_B}{\mathbf{V}} = -[\dot{X}\sin\theta + \dot{Y}\cos\theta]\,\vec{n}_r + [\dot{Y}\sin\theta - \dot{X}\cos\theta]\,\vec{n}_\theta \tag{D.36}$$

The velocity of point S relative to O is given by

$$\overset{R\to S/O_j}{\mathbf{V}} = \overset{R'\to S/O_j}{\mathbf{V}} + {}^R\vec{\omega}^{R'} \times {}_{O_j}\vec{P}^S$$

where

${}^R\omega^j$ = angular velocity of journal in rotating reference frame R'

$\overset{R'\to S/O_j}{\mathbf{V}} = {}^{R'}\omega^j\vec{n}_z \times (R\vec{n}_r)$

$= {}^{R'}\omega^j R\vec{n}'_\theta$

$$\overset{R\to S/O_j}{\mathbf{V}} = (\dot{\phi} + {}^{R'}\omega^j)R\vec{n}'_\theta = {}^R\omega^j R\vec{n}'_\theta \tag{D.37}$$

The total velocity of point S in R is given by

$$\overset{R\to S}{\mathrm{V}} = -[\dot{e}\cos(\theta-\phi)+e\dot{\phi}\sin(\theta-\phi)]\vec{n}_r + [\dot{e}\sin(\theta-\phi)-e\dot{\phi}\cos(\theta-\phi)]\vec{n}_\theta + {}^R\omega^j R\vec{n}'_\theta$$

or

$$\overset{R\to S}{\mathrm{V}} = -[\dot{X}\sin\theta+\dot{Y}\cos\theta]\vec{n}_r + [\dot{Y}\sin\theta - \dot{X}\cos\theta]\vec{n}_\theta + {}^R\omega^j R\vec{n}_\theta \qquad \text{(D.38)}$$

The $\vec{n}'_\theta$ unit vector is given by

$$\vec{n}'_\theta = \cos\gamma\,\vec{n}_\theta + \sin\gamma\,\vec{n}_r \qquad \text{(D.39)}$$

now, consider triangle O_j, O_b, S

$$\sin\gamma = \frac{e\sin(\theta-\phi-\gamma)}{R+C-h}$$

if $C/R \ll 1.0$, then $\gamma \ll 1.0$

Hence

$$\sin\gamma = \gamma \approx \frac{e\sin(\theta-\phi)}{R}$$

Since

$$h = C + e\cos(\theta-\phi)$$

$$\frac{\partial h}{\partial\theta} = -e\sin(\theta-\phi)$$

hence

$$\gamma = \frac{1}{R}\frac{\partial h}{\partial\theta}$$

$$\vec{n}'_\theta = \vec{n}_\theta - \frac{1}{R}\frac{\partial h}{\partial\theta}\vec{n}_r \qquad \text{(D.40)}$$

The total velocity of point S in R is given by

$$\overset{R\to S}{\mathrm{V}} \underset{(\mathrm{D.37,\,D.39})}{=} -\left[\dot{X}\sin\theta+\dot{Y}\cos\theta+{}^R\omega^j\frac{\partial h}{\partial\theta}\right]\vec{n}_r + [\dot{Y}\sin\theta-\dot{X}\cos\theta+R\,{}^R\omega^j]\vec{n}_\theta \qquad \text{(D.41)}$$

$$\overset{R\to S}{\mathbf{V}} \underset{(D.37,\,D.39)}{=} -\left[\dot{e}\cos(\theta-\phi)+e\dot{\phi}\sin(\theta-\phi)+{}^{R}\omega^{j}\frac{\partial h}{\partial\theta}\right]\vec{\mathrm{n}}_r$$

$$+[\dot{e}\sin(\theta-\phi)-e\dot{\phi}\cos(\theta-\phi)+R\omega]\vec{\mathrm{n}}_\theta$$

Differentiating Eq. (D.31)

$$\frac{dh}{dt}=\dot{e}\cos(\theta-\phi)+e\sin(\theta-\phi)\dot{\phi}$$

$$=\dot{e}\cos(\theta-\phi)-\dot{\phi}\frac{\partial h}{\partial\theta} \tag{D.42}$$

$$\therefore \overset{R\to S}{\mathbf{V}} \underset{(D.41,\,D.40)}{=} -\left[\frac{dh}{dt}+{}^{R}\omega^{j}\frac{\partial h}{\partial\theta}\right]\vec{\mathrm{n}}_r+\left[{}^{R}\omega^{j}R-\frac{d}{dt}\left(\frac{\partial h}{\partial\theta}\right)\right]\vec{\mathrm{n}}_\theta \tag{D.43}$$

The velocity of point S in a Newtonian reference frame is given by

$$\overset{N\to S}{\mathbf{V}}=\overset{N\to S/O_b}{\mathbf{V}}+\overset{N\to O_b}{\mathbf{V}}$$

where

$$\overset{N\to O_b}{\mathbf{V}}=\text{velocity of bearing center}$$
$$=\dot{X}_0\vec{\mathrm{n}}_x+\dot{Y}_0\vec{\mathrm{n}}_y$$
$$\overset{N\to S}{\mathbf{V}}=\overset{N\to O_b}{\mathbf{V}}+\overset{R\to S/O_b}{\mathbf{V}}+\overset{N\to R}{\omega}\times\overset{O_b\to S}{\mathbf{P}}$$

where ${}^{N}\omega^{R}=\omega^{b}=$ bearing angular speed in Newtonian reference frame

$$\underset{(D.42)}{=}\dot{X}_0\vec{\mathrm{n}}_x+\dot{Y}_0\vec{\mathrm{n}}_y+[({}^{N}\omega^{R}+{}^{R}\omega^{j})R$$

$$-\frac{d}{dt}\left(\frac{\partial h}{\partial\theta}\right)\Big]\vec{\mathrm{n}}_\theta-\left[\frac{dh}{dt}+{}^{R}\omega^{j}\frac{\partial h}{\partial\theta}\right]\vec{\mathrm{n}}_r \tag{D.44}$$

$$\overset{N\to E}{\mathbf{V}}=\dot{X}_0\vec{\mathrm{n}}_x+\dot{Y}_0\vec{\mathrm{n}}_y+(R+C){}^{N}\omega^{R}\vec{\mathrm{n}}_\theta$$

$$=(\dot{X}_0\cos\theta+\dot{Y}_0\sin\theta)\vec{\mathrm{n}}_r$$

$$+(-\dot{X}_0\sin\theta+\dot{Y}_0\cos\theta+(R+C){}^{N}\omega^{R})\vec{\mathrm{n}}_\theta \tag{D.45}$$

The journal-bearing velocity components are given by

$$U_2=-\dot{X}_0\sin\theta+\dot{Y}_0\cos\theta+(R+C)\omega_b$$

$$U_1=-\dot{X}_0\sin\theta+\dot{X}_0\cos\theta+R\omega_j-\frac{d}{dt}\left(\frac{\partial h}{\partial\theta}\right)$$

$$V_2 = \dot{X}_0 \cos\theta + \dot{Y}_0 \sin\theta$$

$$V_1 = -\left[\frac{dh}{dt} + {}^R\omega^J \frac{\partial h}{\partial \theta}\right]$$

$$W_2 = W_1 = 0$$

Substituting the above velocity components into Eq. (D.28) and after eliminating higher order terms, h/R, C/R, $\dot{X}_0/R\omega$, $\dot{Y}_0/R\omega$ results in the following dimensionless Reynolds equation

$$\frac{\partial}{\partial\theta}\left(PH^3 \frac{\partial P}{\partial\theta}\right) + \left(\frac{R}{L}\right)^2 \frac{\partial}{\partial\eta}\left(PH^3 \frac{\partial P}{\partial\eta}\right) = \Lambda \frac{\partial}{\partial\theta}(PH) + \frac{\sigma\partial}{\partial\tau}(PH) \tag{D.46}$$

where

P = dimensionless pressure $= p/P_a$

P_a = ambient pressure

Λ = compressibility parameter $= \dfrac{6\mu(\omega_j + \omega_b)}{P_a}\left(\dfrac{R}{C}\right)^2$

H = dimensionless film thickness $= h/C$

L = bearing width

η = dimensionless width $= Z/L$

σ = squeeze film number $= \dfrac{12\mu f}{Pa}\left(\dfrac{R}{C}\right)^2$

τ = dimensionless time $= tf$

f = system characteristic frequency

To obtain the form of the Reynolds equation with respect to the rotating reference frame R', we transform the time derivative of pressure as follows

$$\frac{{}^R\partial P}{\partial t} = \frac{{}^{R'}\partial P}{\partial t} + \dot{\phi}\frac{\partial P}{\partial\theta}$$

and substitute the above into Eq. (D.45) to obtain

$$\frac{\partial}{\partial\theta}\left(PH^3 \frac{\partial P}{\partial\theta}\right) + \left(\frac{R}{L}\right)^2 \frac{\partial}{\partial\eta}\left(PH^3 \frac{\partial P}{\partial\eta}\right) = \Lambda\left[2\frac{{}^{R'}\partial}{\partial T'}(PH) + \left(1 - \frac{2\dot{\phi}}{\omega}\right)\frac{\partial(PH)}{\partial\theta}\right] \tag{D.47}$$

where

$H = 1 + \epsilon\cos\theta$

$\omega = \omega_j + \omega_b$

$T' = \omega t$

Examination of Eqs. (D.46) and (D.47) shows that the form of the pressure equation depends upon the system of coordinates used. For example, if the rotor motion is stable synchronous precession in which e is constant (vertical rotor), the time-transient term of Eq. (D.46) expressed in fixed coordinates is nonzero, but the transient pressure term of Eq. (D.47) expressed in the relative reference frame R' is zero.

The influence of compressibility can be neglected only when both the compressibility number Λ and squeeze film number σ approach zero. In this case Eq. (D.47) reduces to

$$\frac{\partial}{\partial\theta}\left(H^3\frac{\partial P}{\partial\theta}\right)+\left(\frac{R}{L}\right)^2\frac{\partial}{\partial\eta}\left(H^3\frac{\partial P}{\partial\eta}\right)=\Lambda\left[\left(1-\frac{2\dot{\phi}}{\omega}\right)\frac{\partial H}{\partial\theta}+\frac{2}{\omega}\dot{H}\right] \quad \text{(D.48)}$$

which represents the governing Reynolds equation for an incompressible fluid.

D.5 BEARING FRICTION

The friction shear stress acting on the rotating journal is given by

$$\tau=\mu\frac{\partial u}{\partial y}\bigg|_{y=h} \quad \text{(D.49)}$$

$$u(y)=\frac{1}{2\mu R}\frac{\partial P}{\partial\theta}\,y(y-h)+\frac{y}{h}\,(R\omega+\dot{e}\sin\theta-\dot{\phi}e\cos\theta)$$

$$\therefore\tau=\frac{h}{2R}\frac{\partial P}{\partial\theta}+\frac{\mu}{h}\,[R\omega-\dot{\phi}e\cos\theta+\dot{e}\sin\theta] \quad \text{(D.50)}$$

The first term in the above expression represents the shear stress contribution due to pressure profile drag and the second term represents the velocity drag. An order of magnitude analysis shows that the pressure profile drag may be neglected in comparison to the velocity drag:

$$\tau\approx\left(\frac{C}{R}\right)\left(\frac{W}{D}\right)+\mu\left(\frac{R}{C}\right)\omega$$

$$\approx\mu\left(\frac{R}{C}\right)\omega\left[1+\left(\frac{C}{R}\right)^2\left(\frac{W}{\mu R\omega}\right)\right]$$

Typical values for a gas bearing are

$D=2$ in. $\qquad \mu=2.51\times10^{-6}$ lb-sec/in.^{2}

$C=0.001$ in. $\qquad \omega=1000$ rad/sec

$W=50$ lb

$$\tau\approx\mu\left(\frac{R}{C}\right)\omega\left[1+\frac{(10^{-6})\,(50)}{(2.51\times10^{-6})1\times10^3}\right]$$

$$\approx f[1+0.02]$$

Reference 28 shows that neglecting the shear contribution of the pressure gradient term will cause at most only a 10-percent error in the friction force

$$\therefore \tau = \mu \frac{R}{C} \frac{\omega}{H} \left[1 + \left(\frac{C}{R} \right) (\dot{\epsilon} \sin \theta - \dot{\phi} \epsilon \cos \theta) \right]$$

Thus since $C/R \approx 1 \times 10^{-3}$, the journal friction is relatively independent of the precession rate. The net shear force component acting normal to the journal-bearing line of centers is given by

$$F_N = \int_0^L \int_0^{2\pi} \mu \frac{\partial u}{\partial y} \cos \theta R d\theta dz$$

$$= \frac{\mu R^2 L \omega}{C} \int_0^{2\pi} \frac{\cos \theta d\theta}{1 + \epsilon \cos \theta} \tag{D.51}$$

To integrate the above expression consider the following. Let

$$z = e^{i\theta} \qquad \cos \theta = \frac{z + z^{-1}}{2}$$

$$d\theta = -idz/z$$

$$\int_0^{2\pi} \frac{d\theta}{1 + \epsilon \cos \theta} = i \oint \frac{2dz}{z[2 + \epsilon(z + z^{-1})]} = \frac{i2}{\epsilon} \oint \frac{dz}{(z - z_1)(z - z_2)}$$

where the roots z_1 and z_2 are given by

$$z_1 = -\frac{1}{\epsilon} \left[1 + \sqrt{1 - \epsilon^2} \right]$$

$$z_2 = -\frac{1}{\epsilon} \left[1 - \sqrt{1 - \epsilon^2} \right]$$

The root z_1 lies inside the unit circle and hence the integral is singular when $z = z_1$. The value of the integral is given by

$$\oint f(z) dz = 2\pi i \, \Sigma \, Res$$

Where the residue for a simple pole is given by

$$(Z - Z_1) f(z) \Big|_{z = z_2} = \frac{\epsilon}{2\sqrt{1 - \epsilon^2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{1 + \epsilon \cos \theta} = 2\pi i \left[-\frac{i2}{\epsilon} \right] \left[\frac{\epsilon}{2\sqrt{1 - \epsilon^2}} \right] = \frac{2\pi}{\sqrt{1 - \epsilon^2}} \tag{D.52}$$

The bearing normal friction force then is given by

$$F_N = \frac{\mu R^2 L\omega}{C} \int_0^{2\pi} \frac{1}{\epsilon} \left[1 - \frac{1}{1+\epsilon \cos \theta}\right] d\theta$$

$$\underset{(D.51,52)}{=} \frac{\mu R^2 L\omega}{C} \frac{2\pi}{\epsilon} \left[1 - \frac{1}{(1-\epsilon^2)^{1/2}}\right] \tag{D.53}$$

Under normal steady-state conditions, the bearing friction force is neglected in comparison to the hydrodynamic bearing force in the determination of the journal equilibrium position. There are special circumstances in which the bearing friction force is not negligible, such as when ϵ approaches unity, then F_N approaches infinity, and when the journal precession rate $\dot{\phi}$ approaches half the rotor speed. In this case the pressure forces approach zero as $\dot{\phi}$ approaches $\omega/2$. Hence the friction force can be of the same order of magnitude as the hydrodynamic forces under these circumstances.

APPENDIXES: