STABILITY ANALYSIS OF TURBOMACHINERY USING CONSTRAINED MODAL ANALYSIS

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ABSTRACT

The following paper deals with the stability analysis of turbomachinery using component modal synthesis. In the problem of complex damped eigenvalue analysis of rotating machinery, there exists the possibility of real positive roots which may cause the rotor to become unstable at a particular threshold speed. The cause of self-excited rotor instability may be attributed to several areas, such as hydrodynamic fluid film bearings and seals, aerodynamic cross-coupling, and internal rotor friction. The instability observed on turbomachinery is usually associated with the lowest system eigenvalue.

Modal analysis has been extensively used in the dynamic analysis of structures and rotating shafts for a number of years. There are a number of modal sets which may be employed in the analysis. These modal sets are normal modes; constrained and rigid body modes; and free-free with rigid body modes. It has been determined that the use of the constrained flexible modes plus rigid body modes produces the exact characteristic equation.

A standard method of damped eigenvalue analysis for a multistation rotor is by the complex matrix transfer method. However, with long connected trains, this procedure runs into numerical difficulty. In the constrained modal method, accurate undamped constrained modes may be calculated using a minicomputer with 100 station rotors. The required constrained planar modes may also be generated by any of the large finite element codes. A numerical procedure has been developed to generate the characteristic polynomial directly for solution of the complex roots or for analysis of stability by the Routh procedure. Gyroscopic effects, as well as bearing cross-coupling may be included in the analysis, and stability may be quickly determined on a 16 bit processor, with accuracy rivaling a main frame computer using the matrix transfer method.

NOMENCLATURE

A_c amplification factor = $1/2\xi$ (dim)

A_i i'th coefficient of characteristic polynomial

[A] 2n x 2n mass and damping matrix

[B] 2n x 2n mass and stiffness matrix

$[B]_k$ k'th Leverrier matrix

b subscript, bearing

c subscript, constraint mode

[C] damping, N-m/s

C_{ij} modal cross coupling coefficient

$\bar{C}$ damping coefficient = $2C/M\omega_c$ (dim)

[D] 2n x 2n dynamic system matrix (dim)

f frequency ratio = $(\omega_r/\omega_c)^2 = 2K_b/K_s$ (dim)

$[I_x]$ n x n transverse moment of inertia matrix, Kg-m^2

$[I_p]$ n x n polar moment of inertia matrix, Kg-m^2

K_b bearing stiffness, N/m

K_s shaft stiffness, N/m

K $2K_b/K_s$ = stiffness ratio (dim)

[K] n x n stiffness matrix, N/m

[M] n x n mass matrix, kg

M_i modal mass, i'th mode, kg

$\bar{M}_{ro}$ normalized modal mass cross-coupling coef. (dim)

n order of system

p real part of complex root, rad/s

q generalized coordinate

q_c generalized constraint coordinate

q_r generalized rigid body coordinate

r subscript-rigid body

s complex root = p+iv, rad/s

{U} general shaft x,y displacement vector

{X} displacement vector, m

λ complex or complex inverse root, rad/s

$\bar{\lambda}$ complex conjugate root

ω natural frequency, rad/s

ω_c constrained natural frequency, rad/s

ω_r rigid body natural frequency, rad/s

Ω normalization factor, rad/s

v imaginary part of complex root, rad/s

{ϕ} i'th normal mode

Λ normalized frequency = λ/ω_c (dim)

{Φ} orthonormal mode

BACKGROUND AND INTRODUCTION

In the analysis of the dynamic characteristics of high speed rotating machinery, such as compressors and gas turbines, it is desirable to determine the damped eigenvalues of the system. The magnitude of the real coefficient of the damped eigenvalue determines the rotor amplification factor of the system. A more serious problem with rotating machinery at high speeds is the occurrence of self-excited whirl motion. Self-excited whirl motion or rotor instability may be caused by such factors as aerodynamic cross-coupling effects of the impellers, labyrinth and fluid film seals, and hydrodynamic journal bearings. In the linearized rotor-bearing system this is indicated by the occurrence of a positive real root.

A major contribution to the field of rotor-bearing stability was presented by Lund in 1974, when he described the complex matrix transfer procedure to calculate the stability characteristics of multi-station turborotors, including generalized linear bearing coefficients. This paper represented a major advancement in the field of stability analysis. There are, however, inherent numerical difficulties associated with the matrix transfer method. In the absence of scaling of the transfer matrices, numerical round-off errors occur on large station systems which generate inaccurate eigenvalues and eigenvectors. This procedure may be somewhat alleviated by using double precision and scaling of the matrices. However, it cannot be completely avoided.

In this paper, a method based on constrained normal modes plus rigid body modes is presented to determine damped eigenvalues. The results of this procedure are compared to the normal modal method for a simplified system. The area of modal analysis is well developed and is extensively employed by structural engineers to simplify the dynamical representation of the system. One of the standard methods of modal analysis is to eliminate the damping or dissipation terms in the equations of motion and solve for the undamped normal modes of the system. By expressing the deflection as a sum of the undamped normal modes, the modal dynamical equations of motion may be generated. One of the typical assumptions in structural dynamics is that the modal damping cross-coupling terms are small and are thus eliminated. In the case of a turborotor with hydrodynamic fluid film bearings or squeeze film dampers, the modal cross-coupling damping terms can never be eliminated. The assumption that the normal modal equations of motion are uncoupled is based upon the approximation that the damping matrix is proportional to the mass or stiffness matrix. In the case of rotating machinery with bearings or seals, this situation never occurs in practice. It is only valid if the damping of the system is extremely light and of the order of only 1 or 2 percent of critical damping. Even retaining all cross coupling modal damping terms, the normal mode approach may not necessarily yield accurate results.

Fig. 1, for example, represents a 72,000 lb. (32,727 Kg) gas turbine with 54 stations. The first two normal modes for this system are shown

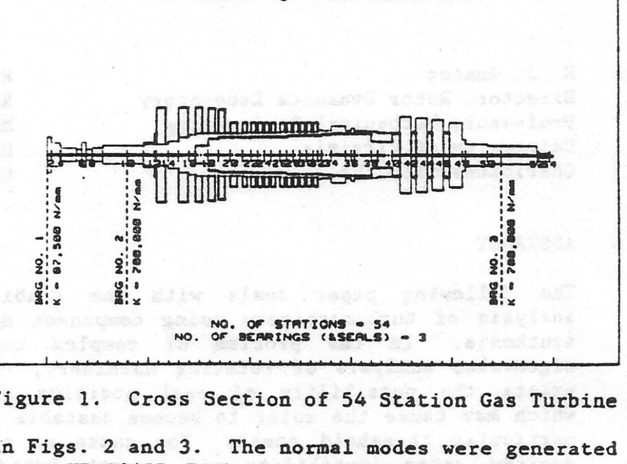

Figure 1. Cross Section of 54 Station Gas Turbine

in Figs. 2 and 3. The normal modes were generated on a HP-9845B Desk Computer with a 16 bit processor. In an attempt to analyze the damped eigenvalues of this system on a main frame computer, numerical difficulties were encountered with the matrix transfer procedure. The analysis of the 70 MW power generation gas turbine as shown in Fig. 1 will be presented in a later paper.

The method of modal analysis appears to be a very attractive procedure to describe the dynamical behavior of such a complex system. The area of modal analysis has received extensive treatment and classical descriptions of this method are given by the various researchers in structural dynamics, such as Hurty and Rubinstein. Modal analysis has been extensively applied to rotating machinery by Bishop, Parkinson, and Black in England and Childs, Nelson, Gunter, Choy and Li, etc. in recent papers in the U.S. This is just to mention a few of the many papers in this area.

The procedure is attractive from the standpoint that the various system modes normally need to be calculated only once. Modes are then used as

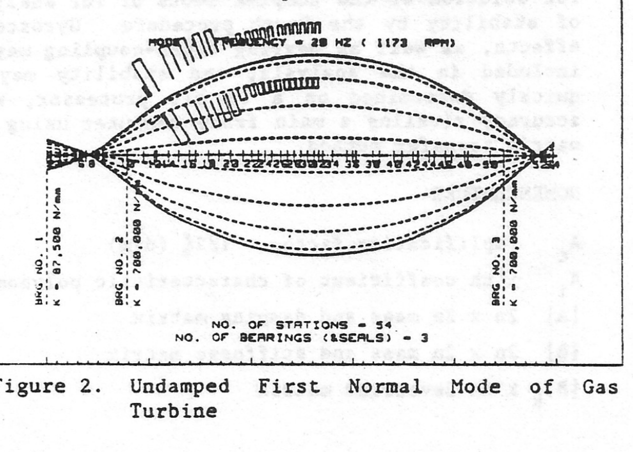

Figure 2. Undamped First Normal Mode of Gas Turbine

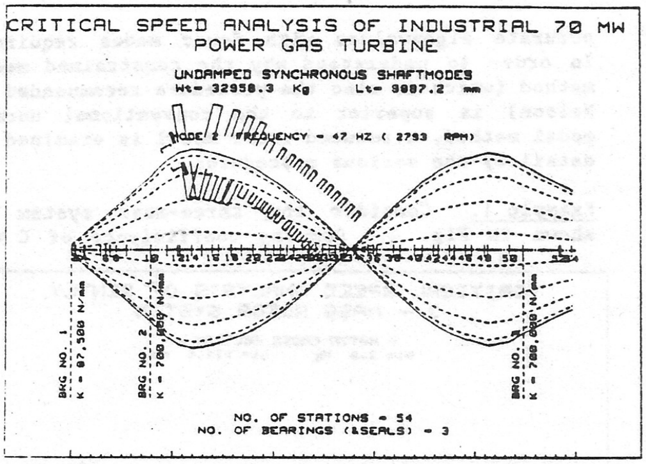

Figure 3. Undamped Second Normal Mode of Gas Turbine

building blocks to describe the generalized equations of motion usually in terms of undamped modes of motion. The required undamped planar constrained modes may also be obtained from standard finite element codes.

There are various sets of mode shapes that may be employed as functional sets to span the vector space of equations. These mode sets may be roughly classified in the catagories of normal, constrained, and free-free rigid body mode sets. Dynamic analysis may also be performed by the use of complex modes as outlined by Foss and also expanded upon by Lund. The undamped modes may be used to generate the complex damped modes. Although a number of papers have been written in various fields on modal analysis, few papers have been written on the errors generated by the truncation to a finite number of modes. Li and Gunter in 1981 presented one study on modal truncation error in component mode analysis of a dual rotor system.

However, considerable research is still required in this area. In this paper it is shown that the use of normal modes may not generate accurate results even if the modal cross coupling damping terms are retained. The use of component modes along with rigid body modes is shown to generate the exact eigenvalues. An excellent presentation on component mode analysis is given by Nelson.

The generation of the damped eigenvalue problem requires the initial solution of the undamped planar problem, assuming the bearings are node points. Once the rotor modal mass and frequencies are obtained, then the rotor elastic properties do not have to be further calculated. The entire stability analysis may be performed on a 16 bit microcomputer with extremely good accuracy.

GENERAL EQUATIONS OF MOTION

The introduction of linear viscous damping into a dynamical system results in the following general matrix formulation:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = F(t) \qquad (1)$$

For a multimass flexible rotor with shaft bow the general equations of motion are of similar form to Eq. (1) as follows:

$$[\bar{M}]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = \{F(t)\} + [K_s]\{U_d\} \qquad (2)$$

where

$$\{U\} = \begin{Bmatrix} (x) \\ (\theta) \\ (y) \\ (\psi) \end{Bmatrix} \qquad \{U_d\} = \begin{Bmatrix} (x_d) \\ (\theta_d) \\ (y_d) \\ (\psi_d) \end{Bmatrix}$$

and

$$[\bar{M}] = \begin{bmatrix} [M] & 0 & 0 & 0 \\ 0 & [I_t] & 0 & 0 \\ 0 & 0 & [M] & 0 \\ 0 & 0 & 0 & [I_t] \end{bmatrix}$$

The damping matrix can be decomposed into the gyroscopic and bearing damping submatrices for future convenience as follows:

$$[C] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & [\omega I_p] \\ 0 & 0 & 0 & 0 \\ 0 & [-\omega I_p] & 0 & 0 \end{bmatrix}_G + \begin{bmatrix} [C_{xx}] & [C_{x\theta}] & [C_{xy}] & [C_{x\psi}] \\ [C_{\theta x}] & [C_{\theta\theta}] & [C_{\theta y}] & [C_{\theta\psi}] \\ [C_{yx}] & [C_{y\theta}] & [C_{yy}] & [C_{y\psi}] \\ [C_{\psi x}] & [C_{\psi\theta}] & [C_{\psi y}] & [C_{\psi\psi}] \end{bmatrix}_B$$

The general stiffness matrix may also be decomposed into two submatrices representing the symmetric shaft stiffness matrix and the linear bearing stiffness matrix.

$$[K] = \begin{bmatrix} [k_{xx}] & [k_{x\theta}] & 0 & 0 \\ [k_{\theta x}] & [k_{\theta\theta}] & 0 & 0 \\ 0 & 0 & [k_{xx}] & [k_{x\theta}] \\ 0 & 0 & [k_{\theta x}] & [k_{\theta\theta}] \end{bmatrix}_S + \begin{bmatrix} [K_{xx}] & [K_{x\theta}] & [K_{xy}] & [K_{x\psi}] \\ [K_{\theta x}] & [K_{\theta\theta}] & [K_{\theta y}] & [K_{\theta\psi}] \\ [K_{yx}] & [K_{y\theta}] & [K_{yy}] & [K_{y\psi}] \\ [K_{\psi x}] & [K_{\psi\theta}] & [K_{\psi y}] & [K_{\psi\psi}] \end{bmatrix}_B$$

The generalized eigenvalue problem for the damped system is formulated by setting F(t) = 0 and assuming the displacement vector {U} to be of the form:

$$\{U\} = \{X\}\, e^{\lambda t} \qquad (3)$$

The generalized eigenvalue problem may be written as:

$$[\lambda^2[M] + \lambda[C] + [K]]\ \{X\} = 0 \qquad (4)$$

Since {X} is in general a nonzero vector, Cramer's rule requires that the determinant of the coefficients must vanish. This leads to the following equation:

$$|\ \lambda^2[M] + \lambda[C] + [K]\ | = 0 \qquad (5)$$

Eq. (5) represents a polynomial of the form

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) - - (\lambda - \lambda_{2n}) \qquad (6)$$

In general the roots λ_i are complex for underdamped systems. Since the [M], [C] and [K] matrices are all real coefficients, the coefficients of the characteristic equation are all real numbers. The complex roots λ_i have a corresponding complex conjugate root $\bar{\lambda}_i$.

For a full system of n degrees of freedom (no zeros in the mass matrix), the order of the polynomial is 2n. In the case where none of the roots are critically damped, the characteristic polynomial is of the form:

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \bar{\lambda}_1)(\lambda - \lambda_2)(\lambda - \bar{\lambda}_2) \cdots (\lambda - \lambda_n)(\lambda - \bar{\lambda}_n) \quad (7)$$

The root λ_i is of the general form:

$$\lambda_i = P_i + iv_i \quad \text{(rad/s)}$$
$$\bar{\lambda}_i = P_i - iv_i \quad (8)$$

The resulting motion corresponding to the ith root is of the form:

$$\{U\} = \{X\}_i e^{p_i t} [\cos v_i t + i \sin v_i t] \quad (9)$$

Hence if the real component P of the complex root λ is greater than zero, the system motion grows exponentially with time and the system is said to be unstable in the linear sense.

From a practical standpoint, it is not desirable to expand out the characteristic polynomial of a large order system as shown in Fig. 1. From the theory of invariants of the characteristic polynomial it is seen that the polynomial may be written in the form:

$$P(\lambda) = \lambda^{2n} + A_1\lambda^{2n+1} + \ldots + A_{2n-1}\lambda + A_{2n} = 0 \quad (10)$$

The coefficients of the characteristic polynomial are called the invariants and

$$A_1 = -\Sigma \lambda_i$$
$$A_{2n} = \prod_{i=1}^{2n} \lambda_i$$

The last coefficient is equal to the product of the eigenvalues, and hence becomes extremely large, even for small order systems. By means of modal analysis and proper scaling, the coefficients of the characteristic polynomial may be kept within bounds.

FORMULATION OF THE DAMPED EIGENVALUE PROBLEM BY NORMAL MODAL ANALYSIS

The general solution of the damped eigenvalue problem as given by Eq. (2) is difficult to evaluate numerically. The complexity of the system may be reduced by using the constrained planar modes along with the rigid body modes. This reduces the multimass system to only a few degrees of freedom in which the normalized characteristic polynomial may be rapidly generated and solved.

Unlike the conventional normal mode method previously employed by Choy and Gunter, and Li and Gunter, the constrained modal method produces more accurate eigenvalues with fewer modes required. In order to understand why the constrained modal method (which is also the procedure recommended by Nelson) is superior to the conventional normal modal method, a reduced rotor model is examined in detail by the various procedures.

<u>Example 1.</u> Consider the three-mass system as shown in Fig. 4. Damping coefficients of C = 1

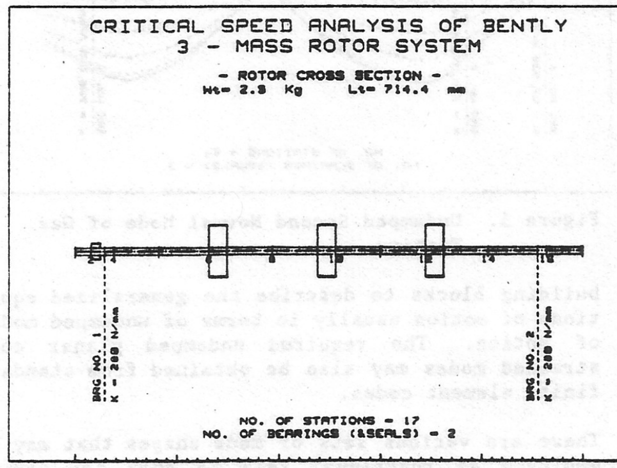

Figure 4. Cross Section of 17 Station Three-Mass Rotor

lb-sec/in. (175 N-sec/in) are applied at each bearing. This 17 station test rotor may be represented by the 5 degree of freedom system as follows:

$$M\lambda^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + C\lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$+ K \begin{bmatrix} 1.215 & -.489 & .348 & -.089 & .015 \\ & 1.326 & -1.274 & .526 & -.089 \\ & & 1.852 & -1.274 & .348 \\ & \text{(SYM)} & & 1.326 & -.489 \\ & & & & 1.215 \end{bmatrix} = 0$$

Where the coefficients M, C, and K are given by:

$M = .00465 \text{ lb} - \text{sec}^2/\text{in}$ (0.815 kg)
$C = 1.00 \text{ lb} - \text{sec/in}$ (175 N-sec/in)
$K = 1{,}000 \text{ lb/in}$ (175,000 N-sec/in)

Table 1 represents three sets of modes obtained for Example 1 with no damping. The first set of modes represent the normal modes with a bearing support spring rate of 1,000 lb/in. (175,000 N/m) located at the bearings as shown in Fig. 4. Fig. 5 represents the first three normal mode shapes for this three-mass model. Figs. 6 and 7 represent the first and second animated mode shapes for the three-mass system. The second set of modes in Table 1 represent the constrained normal modes. These modes are obtained by constraining the motion at the support locations. The third set of mode functions represent the free-free modes. These modes are obtained by assuming no support restraint acts at the end of the station. The first two modes of this set are rigid body modes of zero frequency and the third mode is the system first free-free bending mode.

Table 1
Three-Mass System Mode Shapes and Eigenvalues

I	Undamped Displacement Modes		
FREQ r/min (Hz) rad/s	968.6 (16.1) 101.42	3618 (60.3) 378.85	7911 (131.9) 828.4
M Modal Lb-sec^2/in. (kg)	.0095 (1.666)	.0093 (1.632)	.0091 (1.597)
Station 1	.057	0.334	-0.608
2	0.724	1.000	-0.6906
3	1.000	0.0	1.00
4	0.724	-1.000	-0.6906
5	0.057	-0.334	-0.608

II	Constrained Normal Mode		
FREQ r/min (Hz) rad/s	995 (16.6) 104.17	3947 (66) 413.3	8470 (1412) 886.9
M Modal Lb-sec^2/in. (kg)	.0093 (1.632)	.0093 (1.632)	.0093 (1.632)
Station 2	0.7071	1.00	-0.7071
3	1.000	0.00	1.000
4	0.7071	1.00	-0.7071

III	Free-Free Modes		
FREQ r/min (Hz) rad/s	0	0	4870.5 (81.2) 510
M Modal Lb-sec^2/in. (kg)	.01395 (2.448)	0.0093 (1.632)	.0069 (1.211)
Station 1	1	2	-3.00
2	1	1	-0.50
3	1	0	1.00
4	1	-1	-0.50
5	1	-2	3.00

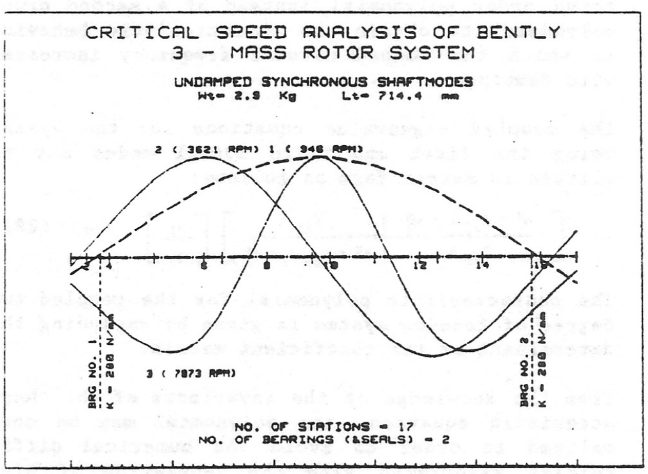

Figure 5. Mode Shapes of Three-Mass Test Rotor

The modes listed in Table 1 are called displacement modes. The maximum value of the displacement mode is unity at a mass station. Since Guyan reduction was used to determine the free-free modes, the displacement at the bearing locations, which are massless, is greater than unity.

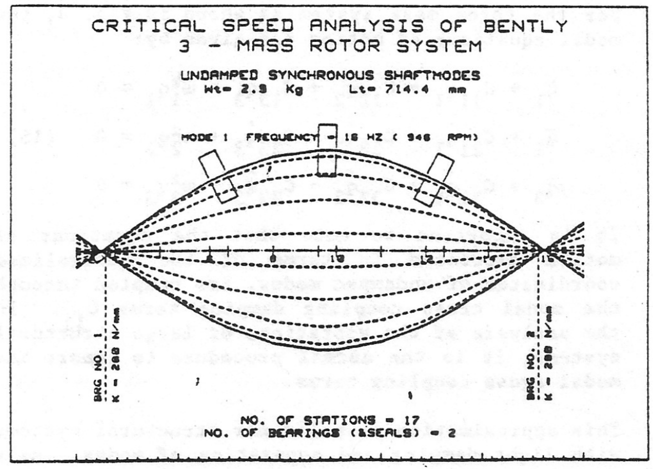

Figure 6. Animated First Mode of Three-Mass Test Rotor

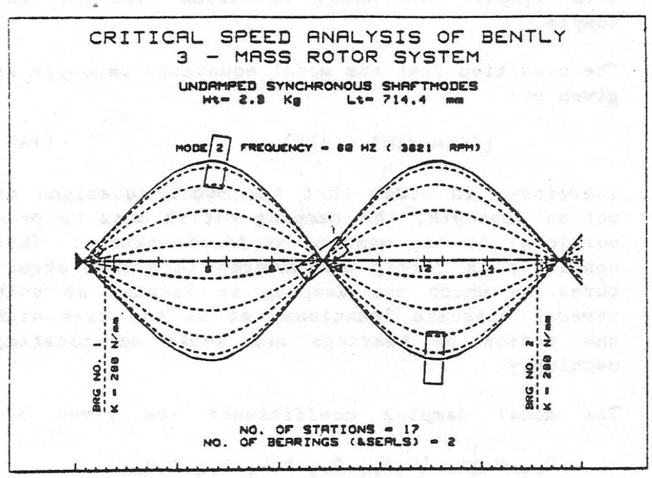

Figure 7. Animated Second Mode of Three-Mass Test Rotor

The system equations of motion are reduced by assuming the displacement vector $\{X\}$ may be represented in terms of the normal modes (I of Table 1) as follows:

$$\{X\} = \sum_{j=1}^{n} q_j \{\phi\}_j \tag{11}$$

Substituting Eq. (11) into Eq. (1) results in:

$$[M] \sum_{j=1}^{n} \ddot{q}_j \{\phi\}_j + [C] \sum_{j=1}^{n} \dot{q}_j \{\phi\}_j + [K] \sum_{j=1}^{n} q_j \{\phi\}_j = F(t) \tag{12}$$

Multiply Eq. (12) by $\{\phi\}_i^T$ and employ the orthogonality conditions that

$$\{\phi\}_i^T [M] \{\phi\}_j = 0 \text{ if } i \neq j; \quad = M_i \text{ if } i = j \tag{13}$$

$$\{\phi\}_i^T [K] \{\phi\}_j = 0 \text{ if } i \neq j; \quad = M_i \omega_i^2 \text{ if } i = j \tag{14}$$

For the three mass system as shown in Fig. 4, the modal equations of motion are given by:

$$\begin{aligned} \ddot{q}_1 + C_{11}\dot{q}_1 + C_{12}\dot{q}_2 + C_{13}\dot{q}_3 + \omega_1^2 q_1 &= 0 \\ \ddot{q}_2 + C_{21}\dot{q}_1 + C_{22}\dot{q}_2 + C_{23}\dot{q}_3 + \omega_2^2 q_2 &= 0 \\ \ddot{q}_3 + C_{31}\dot{q}_1 + C_{32}\dot{q}_2 + C_{33}\dot{q}_3 + \omega_3^2 q_3 &= 0 \end{aligned} \qquad (15)$$

It is important to note that the equations of motion expressed in terms of the generalized coordinates of undamped modes, are coupled through the modal cross coupling damping terms C_{ij}. In the analysis of the vibrations of large structural systems, it is the normal procedure to ignore the modal cross coupling terms.

This approximation is valid for structural systems with light damping and separation of modes. As a general rule, however, if the damping is acting at discrete locations, such as a bearing or a squeeze film damper, the modal equations will not uncouple.

The condition that the modal equations uncouple is given by:

$$[C] = \alpha[M] + \beta[K] \qquad (16)$$

Therefore, in order that the modal equations of motion uncouple, the damping matrix must be proportional to the mass or stiffness matrix. This condition is rarely encountered in actual structures in which the damping is located at only several discrete locations, as is the case with the action of bearings and seals on rotating machinery.

The modal damping coefficients are given by:

$$C_{ij} = \frac{1}{M_i} \left[C_1 \phi_{i1}\, \phi_{j1} + C_5\, \phi_{i5} \cdot \phi_{j5}\right]$$

The modal equations become:

$$\begin{aligned} \ddot{q}_1 + 0.684\, \dot{q}_1 - 7.30\, \dot{q}_3 + 10{,}286\, q_1 &= 0 \\ \ddot{q}_2 + 23.5\, \dot{q}_2 + 143{,}527\, q_2 &= 0 \\ \ddot{q}_3 - 7.30\, \dot{q}_1 + 77.82\, \dot{q}_3 + 686{,}246\, q_3 &= 0 \end{aligned} \qquad (17)$$

Because of symmetry of the modes, the second mode is completely uncoupled from the first and third modes. Hence the second mode of vibration appears to act as a single degree of freedom system. The first and third modal coordinates q_1 and q_3 will be coupled only through the damping matrix. It will be seen that both the first and second damped eigenvalues are in considerable error for moderate values of damping and only approximately correct for small values of damping using the normal mode representation. If the assumption of modal uncoupling is assumed, the results are in considerable error even for small values of damping.

The characteristic polynomial generated by the normal mode method is of 6th order, whereas the characteristic polynomial for the complete system is 8th order. The second mode damped natural frequency actually increases with damping, whereas the modal equations predict the opposite trend.

If the modal cross coupling coefficients are ignored, then the equations of motion are uncoupled and are of the form:

$$\ddot{q}_i + 2\, \xi_i\, \omega_i\, \dot{q}_i + \omega_i^2\, q_i = 0 \qquad (18)$$

and the characteristic polynomial for the system is :

$$\lambda_i^2 + 2\, \xi_i\, \omega_i\, \lambda_i + \omega_i^2 = 0 \qquad (19)$$

Where $\lambda_i = P_i \pm i\, \nu_i$ and

$$P_i = -\, \xi_i\, \omega_i \qquad (20)$$

$$\nu_i = \omega_i \sqrt{1 - \xi_i^2} \qquad (21)$$

For the case where $\xi_i \ll 1$,

$$\nu_i = \omega_i$$

When the damping coefficient ξ_i is much less than 1, then the damped natural frequency ν_i is equal to the undamped natural frequency, ω_i.

Note that for the single degree of freedom system, the damped natural frequency reduces with increasing damping or ξ value. This is not necessarily the case in multidegree of freedom systems such as Eq. (2). Increased damping causes the system eigenvalues to increase, approaching the constrained values in the limit of infinite damping. This is opposite to the behavior of the single degree of freedom system as given by Eq. (21).

It will be seen that it is necessary to have a third order polynomial instead of a second order polynomial to obtain the characteristic behavior in which the damped natural frequency increases with damping.

The coupled eigenvalue equations for the system using the first and third normal modes may be written in matrix form as follows:

$$\left[\begin{array}{c|c} \lambda^2 + C_{11}\,\lambda + \omega_1^2 & C_{13}\,\lambda \\ \hline C_{31}\,\lambda & \lambda^2 + C_{33}\,\lambda + \omega_3^2 \end{array}\right] \left[\begin{array}{c} q_1 \\ \hline q_3 \end{array}\right] = 0 \qquad (22)$$

The characteristic polynomial for the coupled two degree of freedom system is given by expanding the determinant of the coefficient matrix.

From the knowledge of the invariants of the characteristic equation, the polynomial may be normalized in order to avoid the numerical difficulties associated with the generation of high order polynomials.

$$\text{Let } \lambda = \Omega\Lambda, \quad \text{where } \Omega = \sqrt{\omega_1 \omega_3}\,. \qquad (23)$$

The transformed polynomial is now of the form:

$$\Lambda^4 + \bar{A}_3\, \Lambda^3 + \bar{A}_2\, \Lambda^2 + \bar{A}_1\, \Lambda + 1 = 0 \qquad (24)$$

Note that the first and last coefficients of the transformed equation, $\bar{A}_0$ and $\bar{A}_4$ are unity.

Table 2
Influence of Damping on System Using First and Third Normal Modes

Damping	P_1	ν_1	P_3	ν_3
0	0	101.42	0	828.4
1	-.339	101.3	-38.9	827.44
10	-3.37	101.6	-389.1	727.6
50	-13.6	109.5	-154.4	7.4×10^{-10}
100	-12.7	120.1	-62.2	6.6×10^{-10}
1000	-1.54	126.8	-5.57	-4.49×10^{-10}

From Table 2, it is seen that as damping increases, from 1 lb-sec/in. (175 N-s/m) to 1,000 lb-sec/in. (175,000 N-s/m) at the bearings, an optimum value is reached by which maximum damping is achieved for the first mode. This damping value appears to be around C = 50 lb-sec/in (8,750 N-s/m). (It will be shown later that the optimum damping is only 10 lb-sec/in.) However, for the third mode, it is seen that as damping increases, the damped frequency of the third mode diminishes rapidly.

From a physical standpoint, this result is not correct, as the damping increases and approaches ∞, then, the values of P_i should approach zero and the values of the damped frequencies ν_1 and ν_3 should approach the values of the constrained natural frequencies of $\omega_{c1} = 104.2$ and $\omega_{c3} = 886.9$ rad/s. For the case of the first mode, the asymptotic value of ν_1 approaches 126.8 rad/s rather than the value of 104.2. Hence for large values of damping, the use of normal modes is considerably in error for the third mode and only approximately accurate for the first mode. A similar problem also exists with the prediction of the second mode damped natural frequency.

The reason for this discrepancy in the calculation of the damped frequencies by the normal mode procedure is that the characteristic equation for the system is 8th order. Using three normal modes, only a 6th order system can be developed. Hence this characteristic increase in the damped frequency with increasing bearing damping cannot be predicted using only the normal modes. This situation may be remedied by the introduction of two additional rigid body modes.

However, rather than introduce the rigid body modes with normal modes, we shall examine the use of the constrained modes along with the rigid body modes. For example, if the constrained mode is used in conjunction with a rigid body mode, then it can be demonstrated that this system will generate the correct 3rd order characteristic equation for the simple Jeffcott rotor on damped flexible supports.

FORMULATION OF THE DAMPED EIGENVALUE PROBLEM BY CONSTRAINED MODES

The displacements for the sample three-mass system will be given by:

$$\{X\} = \sum_{i=1}^{3} q_{ic}\,\{\phi\}_c + \sum_{i=1}^{2} q_{ir}\,\{\phi\}_r \qquad (25)$$

It is important to note that the constrained modes are not necessarily orthogonal to the rigid body modes. The displacements $\{X\}$ may be expressed in terms of either the displacement mode shapes $\{\phi\}$ or in terms of the orthonormal mode shapes $\{\Phi\}$ where

$$\{\Phi\}_i = \frac{1}{\sqrt{M_i}}\,\{\phi\}_i \qquad (26)$$

Table 3 represents the five orthonormal mode shapes required to describe the three-mass system of Fig. 4. The first three mode shapes listed in Table 3 are constrained modes and the last two are rigid body cylindrical and conical modes.

Table 3
Orthonormal Mode Shapes and Eigenvalues For Three-Mass System

	Constrained Modes			Rigid Body	
Mode No.	1	2	3	4	5
Freq. r/min	995	3,947	8,470	0	0
rad/s	104.17	413.3	886.9		
M-Modal Mass ($Lb\text{-}sec^2/in.$)	.0093	.0093	.0093	0.01395	.0093
Station 1	0	0	0	8.467	20.739
2	7.332	10.370	-7.332	8.467	10.370
3	10.370	0.0	10.370	8.467	0.0
4	7.332	-10.370	-7.332	8.467	-10.370
5	0	0	0	8.467	-20.739

The displacements can be written in general as:

$$\{X\} = \sum_{i=1}^{5} q_i\,\{\Phi\}_i \qquad (27)$$

Applying the above displacement relationship and multiplying by $\{\Phi\}^T$ gives the following set of modal equations for the generalized coordinates q_i:

$$\ddot{q}_1 + M_{14}\,\ddot{q}_4 + M_{15}\,\ddot{q}_5 + \omega_{c1}{}^2 = 0$$
$$\ddot{q}_2 + M_{24}\,\ddot{q}_4 + M_{25}\,\ddot{q}_5 + \omega_{c2}{}^2 = 0$$
$$\ddot{q}_3 + M_{34}\,\ddot{q}_4 + M_{35}\,\ddot{q}_5 + \omega_{c3}{}^2 = 0 \qquad (28)$$
$$\ddot{q}_4 + M_{41}\,\ddot{q}_1 + M_{42}\,\ddot{q}_2 + M_{43}\,\ddot{q}_3 + C_{44}\,\dot{q}_4 + \omega_{r1}{}^2 = 0$$
$$\ddot{q}_5 + M_{51}\,\ddot{q}_1 + M_{52}\,\ddot{q}_2 + M_{53}\,\ddot{q}_3 + C_{55}\,\dot{q}_5 + \omega_{r2}{}^2 = 0$$

It is seen that the mass system is not diagonal and is given by:

$$[M] = \begin{bmatrix} 1.00 & 0 & 0 & 0.9856 & 0 \\ 0 & 1.00 & 0 & 0 & 1.00 \\ 0 & 0 & 1.00 & -0.1691 & 0 \\ 0.9856 & 0 & -0.1691 & 1 & 0 \\ 0 & 1.00 & 0 & 0 & 1 \end{bmatrix} \qquad (29)$$

The coefficients for the damping matrix are given by:

$$[C] = C \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 143.38 & \\ & & & & 215 \end{bmatrix} \qquad (30)$$

The total stiffness matrix of the system is composed of the bearing stiffness matrix plus the stiffness matrix of the shaft corresponding to free-free end conditions:

$$[K] = [\ K\ ]_b + [K]_S \tag{31}$$

The constrained frequencies are given by:

$$\begin{aligned} \omega_{ci}^2 &= \{\Phi\}_{ci}^T\ [[\ K\ ]_b + [K]_S]\ \{\Phi\}_{ci} \\ &= \{\Phi\}_{ci}^T\ [K]_S\ \{\Phi\}_{ci} \end{aligned} \tag{32}$$

In the absence of external bearings acting at the interior points X_2 to X_4. The components ω_{ci}^2 are given by:

$$\omega_{c1}^2 = 104.17^2 = 10{,}851.4$$

$$\omega_{c2}^2 = 413.3^2 = 170{,}816.9$$

$$\omega_{c3}^2 = 886.9^2 = 786.591.6$$

The rigid body frequencies are given by:

$$\begin{aligned} \omega_{ri}^2 &= \{\Phi\}_{ri}^T\ [[\ K\ ]_b + [K]_S]\ \{\Phi\}_{ri} \\ &= \{\Phi\}_{ri}^T\ [\ K\ ]_b\{\Phi\}_{ri} \end{aligned} \tag{33}$$

Since $[K]_S$ is a singular matrix of order 2,

$$\omega_{r1}^2 = \{\phi\}_{r1}^T\ [K]_b\ \{\phi\}_{r1} = 143{,}380 = \frac{2K_b}{3m} \tag{34}$$

$$\omega_{r2}^2 = \{\Phi\}_{r2}^T\ [\ K\ ]_b\ \{\Phi\}_{r2} = 860{,}212 = \frac{4K_b}{m}$$

The modal stiffness matrix is given by:

$$[K]_{modal} = 10^5 \begin{bmatrix} 0.108514 & 0 & 0 & 0 & 0 \\ 0 & 1.7082 & 0 & 0 & 0 \\ 0 & 0 & 7.8659 & 0 & 0 \\ 0 & 0 & 0 & 1.433 & 0 \\ 0 & 0 & 0 & 0 & 8.6021 \end{bmatrix} \tag{35}$$

The diagonal modal [K] matrix may be normalized by dividing the elements by ω_{c1}^2.

$$[\bar{K}]_{modal} = \begin{bmatrix} 1 & & & & \\ & \bar{f}_2 & & & \\ & & \bar{f}_3 & & \\ & & & \bar{f}_4 & \\ & & & & \bar{f}_5 \end{bmatrix} \tag{36}$$

where

$$\bar{f}_i = \left(\frac{\omega_i}{\omega_{c1}}\right)^2 \tag{37}$$

The diagonal damping matrix is normalized by dividing by ω_{c1}. This normalization procedure is equivalent to a dimensionless time transformation of the form $\omega_{c1}\ t = \tau$. The normalized [C] matrix is given by:

$$[\bar{C}] = C \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1.376 & \\ & & & & 2.065 \end{bmatrix} \tag{38}$$

The system modal equations of motion are given by:

$$[\ddot{M}]\ \{q\} + [\ \dot{C}\ ]\ \{q\} + [\ f_i\ ]\ \{q\} = 0 \tag{39}$$

GENERATION OF CHARACTERISTIC EQUATION

The system given as follows,

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = F(t) \tag{40}$$

may be converted into a system of coupled first order equations by:

$$\{y\} = \begin{bmatrix} \{\dot{x}\} \\ \{x\} \end{bmatrix};\ \{\dot{y}\} = \begin{bmatrix} \{\ddot{x}\} \\ \{\dot{x}\} \end{bmatrix}$$

$$\begin{bmatrix} 0 & [M] \\ [M] & [C] \end{bmatrix} \begin{bmatrix} \{\ddot{x}\} \\ \{\dot{x}\} \end{bmatrix} + \begin{bmatrix} [-M] & 0 \\ 0 & [K] \end{bmatrix} \begin{bmatrix} \{\dot{x}\} \\ \{x\} \end{bmatrix} = \begin{bmatrix} 0 \\ \{F(t)\} \end{bmatrix} \tag{41}$$

The form of the 2n order equation in {y} is given by:

$$[A]\ \{\dot{y}\} + [B]\ \{y\} = \{Q\} \tag{42}$$

Consider the homogeneous equation with {Q} = 0.

Let

$$\{y\} = \{Y\}e^{st} \tag{43}$$

$$[D]\{y\} = \lambda\ \{Y\}$$

Where $\lambda = \frac{1}{s}$ inverse complex root

$$[D] = -[B]^{-1}\ [A] \tag{44}$$

$$= -\begin{bmatrix} -[M]^{-1} & 0 \\ 0 & [K]^{-1} \end{bmatrix} \begin{bmatrix} 0 & [M] \\ [M] & [C] \end{bmatrix}$$

$$= \left[\begin{array}{c|c} 0 & I \\ \hline -[K]^{-1}[M] & -[K]^{-1}\ [C] \end{array}\right] \tag{45}$$

The above system may be iterated directly to determine the complex eigenvalues or the characteristic polynomial may be expanded by the application of Leverrier's algorithm. The characteristic polynomial is given by:

$$|\ [D] - \lambda[I]\ | = 0 \tag{46}$$

If the above matrices were used in Leverrier's algorithm, then considerable numerical difficulties would result. The $K^{-1}M$ and $K^{-1}C$ matrices may be scaled as follows:

Let

$$\Omega t = \tau$$

$$\Omega^2[M]\ \{\ddot{x}\} + \Omega[C]\ \{\dot{x}\} + [K]\ \{x\} = 0 \tag{47}$$

$$\Omega^2[K]^{-1}\ [M]\ \{\ddot{x}\} + \Omega[K]^{-1}\ [C]\ \{\dot{x}\} + \{x\} = 0 \tag{48}$$

Let the choice of Ω be the first constrained natural frequency (104 rad/s).

The normalized matrices are given by:

(2) The normal mode set yields only the correct damped eigenvalues for low values of damping. The free-free mode set should be avoided. These particular modes do not form a complete set of functions to properly span the vector space.

(3) The undamped normal modes are generated by setting the damping equal to zero. When these modes are used to express the dynamical equations of motion, the existence of damping caused by bearings or seals will cause modal cross-coupling damping terms to appear which couples the equations of motion. The normal structural procedure is to ignore the modal cross-coupling damping terms by assuming that the damping matrix is proportional to the mass or the stiffness matrix. With real turbomachinery, such a condition never occurs and the equations of motion cannot be considered to be uncoupled unless the modal damping coefficients are extremely low, of the order of two percent of critical damping.

(4) For the case of moderate to high values of bearing damping, the use of the normal modes will not result in the correct eigenvalues, either as to the real part (the damping) or the imaginary part (the damped natural frequency). The prediction of the first damped mode is only moderately accurate, using the normal modes. However, the second and third damped modes are considerably in error for large values of bearing damping. In previous work presented by Li and Gunter, on analysis of gas turbine engine vibrations by the normal mode approach, many higher order modes had to be retained in order to maintain accuracy of the lower mode response.

(5) There is only one set of modes that was found to generate the exact characteristic polynomial. This was the use of the constrained flexible modes plus the addition of two rigid body modes. This particular modal set has the advantage that the constrained normal modes are obtained by specifying 0 bearing displacements. Therefore, the constrained modes are independent of bearing stiffness.

(6) In this paper, it was shown how the exact characteristic polynomial may be generated by the use of Leverrier's algorithm. By means of this algorithm, the coefficients of the characteristic polynomial may be rapidly determined. These characteristic coefficients may be examined by Routh's criteria for stability, or the characteristic equation may be solved directly. Normally, one should not attempt to generate the characteristic polynomial for large order systems, as the coefficients of the polynomial become increasingly large. In this paper, it is shown how a simple scaling procedure may be incorporated with the generation of the characteristic polynomial in order to keep the coefficients within bounds. The scaling procedure was found to be most successful and coefficients for twentieth-order polynomials can be easily generated. This procedure has been adapted to the general stability analysis of turborotors, including eight bearing stiffness and damping coefficients per bearing and shaft gyroscopics.

(7) The constrained modal method may be readily applied to the stability analysis of turborotors. The required planar modes may be generated by a standard finite element program.

(8) The accuracy of the constrained modal method is superior to the normal modal method. In many cases only 1 constrained and 2 rigid body modes are necessary for stability calculations.

(9) The computer run time using the constrained modal method is two orders of magnitude faster than the complex matrix transfer method.

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$$[K]^{-1}[M] = \begin{bmatrix} 0.0000 & .0377 & .0251 & .0126 & 0.0000 \\ 0.0000 & .3163 & .3739 & .2409 & 0.0000 \\ 0.0000 & .3739 & .5321 & .3739 & 0.0000 \\ 0.0000 & .2409 & .3739 & .3163 & 0.0000 \\ 0.0000 & .0126 & .0251 & .0377 & 0.0000 \end{bmatrix}$$

$$[K]^{-1}[C] = \begin{bmatrix} 1.0400 & 0.0000 & 0.0000 & 0.0000 & .0000 \\ .7800 & 0.0000 & 0.0000 & 0.0000 & .2600 \\ .5200 & 0.0000 & 0.0000 & 0.0000 & .5200 \\ .2600 & 0.0000 & 0.0000 & 0.0000 & .7800 \\ .0000 & 0.0000 & 0.0000 & 0.0000 & 1.0400 \end{bmatrix}$$

The assembled dynamic matrix D is given by:

$$[D] = \left[\begin{array}{ccccc|ccccc} 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \\ \hline 0.0000 & -.0377 & -.0251 & -.0126 & 0.0000 & -1.0400 & 0.0000 & 0.0000 & 0.0000 & -.0000 \\ 0.0000 & -.3163 & -.3739 & -.2409 & 0.0000 & -.7800 & 0.0000 & 0.0000 & 0.0000 & -.2600 \\ 0.0000 & -.3739 & -.5321 & -.3739 & 0.0000 & -.5200 & 0.0000 & 0.0000 & 0.0000 & -.5200 \\ 0.0000 & -.2409 & -.3739 & -.3163 & 0.0000 & -.2600 & 0.0000 & 0.0000 & 0.0000 & -.7800 \\ 0.0000 & -.0126 & -.0251 & -.0377 & 0.0000 & -.0000 & 0.0000 & 0.0000 & 0.0000 & -1.0400 \end{array}\right]$$

Let the characteristic equation be expressed in the form:

$$\Lambda^n + A_1 \Lambda^{n-1} + A_2 \Lambda^{n-2} + A_n = 0 \qquad (49)$$

The coefficients of the characteristic equation may be determined by Leverrier's algorithm as follows:

$$A_1 = -\text{trace } [D] \qquad (50)$$

$$[B]_1 = [D] + A_1 [I]$$

$$A_2 = -\tfrac{1}{2} \text{ trace } [[D][B]_1]$$

or in general,

$$A_K = -\frac{1}{K} \text{ trace } [[D][B_{K-1}]]; \; K>1 \qquad (51)$$

and

$$[B]_K = [D][B_{K-1}] + A_K [I] \qquad (52)$$

Using Leverrier's algorithm, an eighth order polynomial is generated. Table 4 represents the damped roots for the 3 mass system with the damping varied from 0.1 to 100 lb-sec/in. The roots generated by expanding out the complete M, K and C matrices were identical (within numerical bounds) to the roots generated based on the use of constrained and rigid body modes. Since the 5 modes used form a complete set, the exact characteristic polynomial is generated. The exact solution for

Table 4
Damped Roots of Three-Mass Rotor Systems

C	P_1	V_1	A_1	P_2	V_2	A_2	P_3	V_3	A_3
0.1	-.03	100.4	1453.0	-1.19	378.7	158.6	-4.05	828.3	102.4
1.0	-.342	100.4	146.6	-11.00	382.0	17.3	-28.4	847.4	14.9
2.0	-.67	100.5	75.1	-17.51	389.8	11.1	-28.02	868.1	15.5
5.0	-1.43	101.0	35.4	-15.98	406.4	12.7	-14.79	882.7	29.8
7.0	-1.71	101.5	29.8	-12.71	410.1	16.1	-10.87	884.5	40.7
10.0	-1.84	102.1	27.7	-9.44	412.4	21.8	-7.72	885.4	57.3
12.0	-1.82	102.5	28.2	-8.00	413.1	25.8	-6.46	885.7	68.5
15.0	-1.71	102.9	30.1	-6.50	413.7	31.8	-5.19	886.0	85.4
20.0	-1.48	103.3	34.9	-4.93	414.2*	42.0	-3.90	886.2	113.5
50.0	-.71	103.9	73.5	-2.00	414.7*	103.9	-1.55	886.4	285.7
100.0	-.36	104.0	143.2	-.89	413.7	231.7	*	*	*

the first mode was generated also by using only the first constrained and the first rigid body mode. In comparison with Table 2, the first damped mode based on the undamped normal modes is only accurate for small ranges of damping and the third mode is completely in error.

From Table 4 it is of interest to note that as the damping is increased, the damped natural frequencies increase from the undamped normal mode and approach the constrained normal mode values. It is also of importance to note that the optimum damping for all three modes is not identical. This fact is important when designing a squeeze film bearing for a gas turbine which must operate through multiple modes. For example, with the three-mass test rotor, the optimum damping for the second and third modes is between 1 to 2 Lb-sec/in. (175 to 350 N-s/m) damping, while the optimum damping for the first mode is 10 Lb-sec/in. (1750 N-s/m). Even with the inclusion of optimum damping for the first mode, the lowest possible amplification factor is given as 27.7.

CONTRAINED MODAL ANALYSIS APPLIED TO TURBOROTORS

The gas turbine rotor as shown in Fig. 1 is considerably more complicated than the simplified 3 mass system previously illustrated. The major extensions are the influence of rotor gyroscopic moments and bearing cross coupling coefficients. Both of these effects cause the rotor X-Y directions to be coupled.

This extension to the general case may be readily accomplished by the constrained modal method.

Let

$$\begin{Bmatrix} X \\ \theta \end{Bmatrix} = \sum_{c=1}^{n} q_{xc} \{\phi_c\} + \sum_{r=1}^{2} q_{xr} \{\phi_r\}$$

$$\begin{Bmatrix} Y \\ \psi \end{Bmatrix} = \sum_{c=1}^{n} q_{yc} \{\phi_i\} + \sum_{r=1}^{2} q_{yr} \{\phi_r\} \qquad (53)$$

The modal equations are of the form:

$$\left[\begin{array}{cc|cc} I & 0 & M_{rc} & 0 \\ 0 & I & 0 & M_{rc} \\ \hline M_{rc} & 0 & M & 0 \\ 0 & M_{rc} & 0 & M \end{array}\right] \begin{Bmatrix} \{\ddot{q}_{cx}\} \\ \{\ddot{q}_{cx}\} \\ \{\ddot{q}_{ry}\} \\ \{\ddot{q}_{rx}\} \end{Bmatrix} + \left[\begin{array}{cc|cc} 0 & \omega \bar{I}_p & 0 & 0 \\ -\omega \bar{I}_p & 0 & 0 & 0 \\ \hline 0 & 0 & \bar{C}_{xx} & \bar{C}_{xy} \\ 0 & 0 & \bar{C}_{yx} & \bar{C}_{yy} \end{array}\right] \begin{Bmatrix} \{\dot{q}_{cx}\} \\ \{\dot{q}_{cy}\} \\ \{\dot{q}_{rx}\} \\ \{\dot{q}_{rx}\} \end{Bmatrix}$$

$$+ \left[\begin{array}{cc|cc} \omega_i^2 & & 0 & 0 \\ & \omega_i^2 & 0 & 0 \\ \hline 0 & 0 & \bar{K}_{xy} & \bar{K}_{xy} \\ 0 & 0 & \bar{K}_{yx} & \bar{K}_{yy} \end{array}\right] \begin{Bmatrix} \{q_{cx}\} \\ \{q_{cy}\} \\ \{q_{rx}\} \\ \{q_{ry}\} \end{Bmatrix} = 0 \qquad (54)$$

For the study of rotor stability, it is usually only necessary to retain the first constrained flexible planar mode and the two rigid body modes. The resulting stability calculations are greatly superior to the normal modal method employing 3 or 4 flexible modes. Note that the rotor gyroscopic terms enter as skew symmetric terms in the damping matrix.

SUMMARY AND CONCLUSIONS

(1) There are three sets of undamped modes of motion that may be used as building blocks or modal sets to determine the system damped eigenvalues or forced response. These modes are called the normal modes, the constrained normal modes, and the rigid body free-free modes.

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