The Influence of Fluid Inertia on the Dynamic Properties of Journal Bearings

Based on a first-order perturbation solution in a modified Reynolds number analysis it is presented to determine the effect of the fluid film inertial forces on the dynamic properties of a journal bearing. The corrections to the regular amplitude and velocity coefficients are found to be small, but the accompanying acceleration coefficients which correspond to a virtual mass of several times the mass of the journal itself, could become significant for short times. Numerical results are given in graphical form with dimensional coefficients as functions of the operating eccentricity ratio.

Introduction

Consistent with the assumptions inherent in reducing the Navier-Stokes equations to the Reynolds equation, the conventional laminar, thin film lubrication theory ignores the inertial forces in the fluid film [1, 2]. This is theoretically justified for small values of the Reynolds number (of the order of 1). On the other hand, the assumption of laminar flow cannot be valid when there is a transition to either Taylor vortex flow or to turbulent flow which, for journal bearings of typical dimensions, occurs at a Reynolds number value of approximately 1000 to 3000. Thus, there is an intermediate range, say for values of Reynolds number of the order of 10^6, where inertial effects may become noticeable without strictly justifying the assumption of laminar flow. Several investigators have examined the problem [1, 3, 4, 5, 6, 7, 8, 9] and found the effect to be small. The investigations, however, are usually restricted to bearings operating under steady-state conditions while dynamic conditions have only been considered for simplified bearing geometries [7, 8, 9]. It is the purpose of the present paper to extend the analysis to one of the more common bearing geometries, namely the cylindrical journal bearing, and to consider the influence of the inertial terms on the dynamic properties. Based on the previous findings that show the inertial forces to be small, the analysis employs a first order perturbation expansion in Reynolds number as also done in [9].

Apart from a qualitative evaluation of the effect of the inertial forces on the dynamic bearing properties, the primary reason for undertaking the investigation is to determine the virtual mass coefficients which could be significant in certain applications, for example short times. This is pointed out by Smith [10] whose analysis, however, is approximate and based on intuitive reasoning.

The Governing Equations

With the usual assumptions of thin film lubrication theory the governing differential equations for a fluid element are [1, 2]:

\[
\begin{align*}
\frac{d}{dt} \left( \frac{p}{2 \rho} \right) + \frac{d}{dy} \left( \frac{\rho u^2}{2} + \frac{\rho w^2}{2} \right) &= - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( \rho \frac{\partial u}{\partial y} \right) \\
0 &= - \frac{\partial p}{\partial y} \\
\frac{d}{dt} \left( \frac{\rho u w}{2} + \frac{\rho w^3}{2} \right) + \frac{d}{dy} \left( \frac{\rho u^2 w}{2} + \frac{\rho w^2 y}{2} \right) &= - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( \rho \frac{\partial u w}{\partial y} \right) \\
0 &= - \frac{\partial p}{\partial y} \\
\frac{d}{dy} \left( \frac{\rho u w}{2} + \frac{\rho w^3}{2} \right) &= 0
\end{align*}
\]

The left-hand sides in equations (1) and (3) are the contributions from the inertial forces which disappear in classical lubrication theory, \( u \) and \( w \) are the fluid velocity components in the \( x \) and \( y \) directions, respectively, where \( y \) is the coordinate across the thickness of the film. The fluid pressure is \( p \), the viscosity \( \mu \) and the density \( \rho \).

When the journal, with radius \( R \), rotates with angular speed \( \omega \) and \( x \) is the circumferential coordinate, the velocity boundary conditions become:

\[
\begin{align*}
y &= 0: u = v = w = 0 \\
y &= R: u = R \omega \\
v &= \frac{R}{R} \frac{dy}{dt} + \frac{R}{2} \frac{dR}{dt} \\
w &= 0
\end{align*}
\]

where \( R \) is the local film thickness.

The equations are normalized by means of the dimensionless

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quantities:
\[ \theta = \frac{c}{R}, \quad \xi = \frac{r}{R}, \quad \eta = \frac{\rho}{R}, \quad \tau = \omega t \]
\[ \tilde{w} = \frac{w}{R}, \quad \tilde{W} = \frac{W}{R}, \quad \tilde{U} = \frac{U}{R}, \quad \tilde{V} = \frac{V}{R} \]
\[ \rho = \frac{\nu}{\nu_0 R^2 / \delta} \]
\[ \tilde{b} = \frac{b}{c} \]
where \( c \) is the radial journal bearing clearance. In addition, a parameter \( \lambda \) is required to account for the inertia forces:
\[ \lambda = \frac{\delta}{R} \]

where
\[ \text{Reynolds number } : \text{Re} = \frac{pd}{\nu R} \]

Thus, equations (1)-(6) can be written in dimensionless form as:
\[ \frac{\partial \tilde{u}}{\partial \tilde{\xi}} - \frac{\tilde{u} \tilde{w}}{\tilde{\eta}} - \frac{\tilde{u} \tilde{\xi}}{\tilde{\eta}} + \frac{\partial \tilde{u}}{\partial \tilde{\eta}} = - \frac{\partial \tilde{b}}{\partial \tilde{\xi}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{\eta}^2} \]
\[ \frac{\partial \tilde{v}}{\partial \tilde{\xi}} + \frac{\tilde{w} \tilde{v}}{\tilde{\eta}} - \frac{\tilde{w} \tilde{\xi}}{\tilde{\eta}} + \frac{\partial \tilde{v}}{\partial \tilde{\eta}} = \frac{\partial \tilde{b}}{\partial \tilde{\xi}} + \frac{\partial^2 \tilde{v}}{\partial \tilde{\eta}^2} \]
\[ \text{At the last equation, equation (12), only expresses that the pressure does not vary across the film, and is of no further interest. From equations (13) and (14) it is seen that the inertia forces are of the order \( \xi / \eta \) when the Reynolds number is small (of the order 1), and it is on this basis that these terms are left out in the classical lubrication theory. At moderate values of the Reynolds number, however, the influence of the inertia terms may become noticeable. To investigate the effect, a first-order perturbation solution is carried out where the nonlinear equations are linearized by setting [9]:}
\[ \tilde{b} = \tilde{b}^{(0)} + \tilde{b}^{(1)} + \alpha \tilde{b}^{(2)} \]
\[ \text{and analogously for \( \tilde{u} \) and \( \tilde{v} \). Substituting these expressions into equations (13)-(15) and collecting terms of like order in \( \alpha \) results in the following zero-order equations:}
\[ \frac{\partial \tilde{u}^{(0)}}{\partial \tilde{\xi}} = \frac{\partial^2 \tilde{u}^{(0)}}{\partial \tilde{\eta}^2} \]
\[ \frac{\partial \tilde{v}^{(0)}}{\partial \tilde{\xi}} = \frac{\partial^2 \tilde{v}^{(0)}}{\partial \tilde{\eta}^2} \]
\[ \frac{\partial \tilde{b}^{(0)}}{\partial \tilde{\xi}} = \frac{\partial^2 \tilde{b}^{(0)}}{\partial \tilde{\eta}^2} = 0 \]
\[ \text{with the boundary conditions:}
\[ \eta = 0 : \tilde{u}^{(0)} = \tilde{v}^{(0)} = \tilde{w}^{(0)} = 0 \]
\[ \eta = 1 : \tilde{u}^{(0)} = 1, \tilde{v}^{(0)} = 0; \tilde{w}^{(0)} = \frac{\partial \tilde{b}^{(0)}}{\partial \tilde{\eta}} \]
\[ \text{Integrating equations (20) and (21) twice yields the well-known velocity profile of lubrication theory:}
\[ \frac{\text{Fig. 1} \text{ Correction to the Inverse Sommerlad number.}}{\text{Fig. 2} \text{ Correction to the direct amplitude coefficient } k.} \]
\[ k(x) = \frac{1}{2} \left( 1 + \frac{x}{x_0} \right) \]
\[ k(x) = \frac{1}{2} \left( 1 + \frac{x}{x_0} \right) \]
\[ x_0 = \frac{R}{c} \]
\[ k(x) = \frac{1}{2} \left( 1 + \frac{x}{x_0} \right) \]
\[ x_0 = \frac{R}{c} \]
The first order perturbation equations, akin to equations (23)-(25), are:

\[
\frac{\partial^2 w^{(1)}}{\partial y^2} + \frac{\partial^2 w^{(1)}}{\partial z^2} + \frac{\partial w^{(1)}}{\partial y} + \frac{\partial w^{(1)}}{\partial z} = 0
\]

\[
\frac{\partial w^{(1)}}{\partial y} = \frac{\partial w^{(1)}}{\partial z} = 0
\]

with the boundary conditions:

\[
\eta = 0 \text{ and } \eta = 1 \Rightarrow w^{(1)} = w^{(1)} = 0 \quad (29)
\]

As \( p^{(1)} \) is independent of \( \alpha \), equation (29), equations (7) and (8) may be integrated twice, making use of equations (24), (35) and (36), to calculate \( w^{(1)} \) and \( w^{(1)} \). The boundary conditions, equation (29), together with equation (29) can be applied to derive the first-order equivalent of Reynolds equation. After rather extensive algebraic operations the final equation becomes:

\[
\frac{\partial^2 w^{(1)}}{\partial y^2} + \frac{\partial^2 w^{(1)}}{\partial z^2} + \frac{\partial w^{(1)}}{\partial y} + \frac{\partial w^{(1)}}{\partial z} = \frac{2}{\eta} \left( 3 \frac{\partial^2 s^{(1)}}{\partial y^2} + \frac{\partial s^{(1)}}{\partial y} \right) + \frac{2}{\eta} \left( 3 \frac{\partial^2 s^{(1)}}{\partial z^2} + \frac{\partial s^{(1)}}{\partial z} \right) - \frac{2}{\eta} \left( 3 \frac{\partial^2 s^{(1)}}{\partial y^2} + \frac{\partial s^{(1)}}{\partial y} \right) - \frac{2}{\eta} \left( 3 \frac{\partial^2 s^{(1)}}{\partial z^2} + \frac{\partial s^{(1)}}{\partial z} \right) = \frac{2}{\eta} \left( 3 \frac{\partial^2 s^{(1)}}{\partial y^2} + \frac{\partial s^{(1)}}{\partial y} \right) + \frac{2}{\eta} \left( 3 \frac{\partial^2 s^{(1)}}{\partial z^2} + \frac{\partial s^{(1)}}{\partial z} \right) - \frac{2}{\eta} \left( 3 \frac{\partial^2 s^{(1)}}{\partial y^2} + \frac{\partial s^{(1)}}{\partial y} \right) - \frac{2}{\eta} \left( 3 \frac{\partial^2 s^{(1)}}{\partial z^2} + \frac{\partial s^{(1)}}{\partial z} \right)
\]

The Dynamic Coefficients

The film-thickness is a function of the journal center position. Introducing an \( x-y \)-coordinate system with origin in the bearing center and the \( x \)-axis in the direction of the applied static load, the journal center has the coordinates \((x, y)\) under static conditions. The coordinates are normalized with respect to the radial clearances. Superimposing a dynamic motion with amplitudes \( \Delta x \) and \( \Delta y \), the coordinate of the journal center will be \((x + \Delta x, y + \Delta y)\). The film-thickness will then take the form:

\[
\delta^{(1)}(x + \Delta x, y + \Delta y) = \delta^{(1)}(x, y) + \frac{\partial \delta^{(1)}}{\partial x} \Delta x + \frac{\partial \delta^{(1)}}{\partial y} \Delta y
\]

where \( \frac{\partial \delta^{(1)}}{\partial x} \) and \( \frac{\partial \delta^{(1)}}{\partial y} \) are the partial derivatives of the first-order film-thickness with respect to \( x \) and \( y \), respectively.
and $\Delta t$ around this equilibrium position, the dimensionless local film thickness can be written as:

$$\Delta h = \Delta h_0 + \Delta h_1 \cos \theta + \Delta h_2 \sin \theta$$  \hspace{1cm} (32)

where:

$$\Delta h_0 = 1 + \Delta h_0 \cos \theta + \Delta h_0 \sin \theta$$  \hspace{1cm} (33)

and where the coordinate angle $\theta$ is measured from the negative $z$-axis. Assuming the amplitudes $\Delta h_1$ and $\Delta h_2$ to be small the pressure may be expanded in a first order perturbation such that equation (18) becomes:

$$\bar{p} = \bar{p}_0 + \bar{p}_1 \Delta h_1 + \bar{p}_1 \Delta h_2 \Delta z + \bar{p}_1 \Delta h_1 \Delta z + \bar{p}_1 \Delta h_2 \Delta z + \bar{p}_1 \Delta h_1 \Delta z + \bar{p}_1 \Delta h_2 \Delta z$$  \hspace{1cm} (34)

Similarly, a first order expansion of equation (32) yields:

$$\Delta h^* = \Delta h_0^* \cos \theta + \Delta h_0^* \sin \theta$$  \hspace{1cm} (35)

Substitution of these equations into equations (26) and (31) and collecting terms according to the perturbation variable, results in 11 equations to determine the 12 quantities, $\bar{p}_0, \bar{p}_1, \bar{p}_2, \ldots$ etc.

**Fig. 7** Correction to the direct velocity coefficient $B_{13}$

With $\bar{p}_0, \bar{p}_1, \bar{p}_2, \ldots$ representing the general quantity these equations are of the form:

$$\frac{\partial}{\partial t} (\bar{p}_0 + \bar{p}_1 \Delta t + \bar{p}_2 \Delta t^2 + \cdots) = -\text{RES}^{(1)}$$  \hspace{1cm} (36)

where the right-hand side, RES$^{(1)}$, depends on $\bar{p}_0$ or $\bar{p}_1$ and their first order derivatives (second order derivatives can be eliminated by means of the governing equations for $\bar{p}_0$ or $\bar{p}_1$). As the expressions are quite lengthy they are omitted.

Allowing for film rupture, the adopted boundary condition are:

$$\bar{p} = \frac{x}{L/D}; \bar{p} = 0$$  \hspace{1cm} (37)

$$\bar{p} = \bar{p}_0(0), \quad \bar{p} = \bar{p}_0(D); \quad \bar{p} = \bar{p}_0(D) = 0$$  \hspace{1cm} (38)

where $\Delta t$ indicates the direction of the normal to the boundary curve. To implement the last condition, consider a point $(x_0, z_0)$ on the free boundary under static conditions. Owing to a pressure perturbation $\Delta \bar{p}$, this point moves to the increment $(\Delta x, \Delta z)$ to a point $(x = x_0 + \Delta x, z = z_0 + \Delta z)$. Requiring the pressure to be zero on the new boundary curve, a first order expansion yields

$$\bar{p}(x_0 + \Delta x, z_0 + \Delta z) = 0 = \bar{p}(x_0, z_0) + \sigma \frac{\partial \bar{p}}{\partial x} \Delta x + \frac{\partial \bar{p}}{\partial z} \Delta z$$  \hspace{1cm} (39)

With $\rho = \rho_0 + \Delta \rho$ and retaining only first order terms, this equation reduces to:

$$\bar{p}(x, z) = 0 = \bar{p}(x_0, z_0) + \Delta \bar{p} \sigma \frac{\partial \bar{p}}{\partial x} \Delta x + \frac{\partial \bar{p}}{\partial z} \Delta z$$  \hspace{1cm} (39)

Because $\bar{p}_0(0, z_0) = 0 = \bar{p}_0(D, z_0) = 0 = \bar{p}_0(0, z_0) = 0$, the boundary condition for the pressure perturbation becomes: $\Delta \bar{p}(x_0, z_0) = 0$. With the boundary conditions established, equation (37) can be written in finite difference form and solved numerically (1, 14).

By integrating the film pressures the reaction forces are obtained as:

$$F_0 = \int_{x_0}^{x_f} \bar{p} \left( \frac{\rho_0}{\rho_n} \right) \rho_n \Delta x$$

or in dimensionless form:

$$\bar{F}_0 = \frac{F_0}{\rho_n N d (x_f - x_0)} = \frac{\rho_0}{\rho_n} \frac{L^2}{D} \frac{1}{L/D} \Delta x \bar{p} \left( \frac{\rho_0}{\rho_n} \right) \rho_n \Delta x$$

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where $N$ is the rotational speed in rps ($N = \omega / 2\pi$). Substituting for $p$ from equation (30) the perturbation expansion of the reaction forces become:

$$F_x = F_x^{(0)} + \lambda F_x^{(1)} + \frac{N}{E} F_x^{(2)}$$

and similarly for $F_y$. The relationship between the dimensional and the dimensionless coefficients are:

$$F_x = \frac{1}{2} \bar{N} \bar{D} L \bar{R} \bar{c}/C$$

and similarly for the other coefficients. $S$ is the Sommerfeld number.

$$N = (1 + \lambda \Delta F_x^{(0)}) F_x$$

where $\Delta F_x^{(0)}$ is the static load such that:

$$\Delta F_x^{(0)} = \frac{W}{\pi D L R / C}$$

and similarly for $F_y$. The acceleration coefficients in addition to the usual amplitude and velocity coefficients, equation (30) shows that, when inertia forces are considered, the dynamic fluid film reaction forces also depend on the journal accelerations [10]. The 4 acceleration coefficients, $C_{ax}, C_{ay}, C_{cx}, C_{cy}$ are computed from equation (30) where:

$$C_{ax} = \frac{5}{12} \int_{0}^{\lambda} \frac{1}{D^3} \left( \frac{p \Delta x}{D} \right)^{\frac{1}{2}} \left( \frac{L}{D} \right)^{\frac{1}{2}} \frac{\cos \theta}{D} \, d\theta$$

and similarly for $C_{ay}, C_{cx}, C_{cy}$. The perturbations $p_{ax}$ and $p_{ay}$ are determined from equation (56) in the following form:
where \( p_{r,0} \) and \( p_{r,1} \) are found from:

\[
\frac{\partial^2}{\partial z^2} \left( \frac{1}{\rho_0} \left( \frac{1}{p_{r,0}} \right) \right) + \frac{\partial}{\partial z} \left( \frac{1}{\rho_0} \frac{\partial p_{r,0}}{\partial z} \right) = \frac{6 \Omega \zeta}{\rho_0} \left( \frac{1}{\rho_0} \right) \frac{\cos \theta}{\sin \theta} \left( \frac{\rho_0}{\rho_0} \right)
\]

\[
\frac{\rho_0}{2} \frac{\partial^2 p_{r,1}}{\partial z^2} + \frac{\rho_0}{2} \frac{\partial^2 p_{r,1}}{\partial z^2} = \frac{6 \Omega \zeta}{\rho_0} \left( \frac{1}{\rho_0} \right) \frac{\cos \theta}{\sin \theta} \left( \frac{\rho_0}{\rho_0} \right)
\]

In the special case of concentric journal, \( \tau_z = \tau_s = 0 \), the solution becomes:

\[
\frac{\partial^2}{\partial z^2} \left( \frac{1}{\rho_0} \left( \frac{1}{p_{r,0}} \right) \right) = \frac{6 \Omega \zeta}{\rho_0} \left( \frac{1}{\rho_0} \right) \frac{\cos \theta}{\sin \theta} \left( \frac{\rho_0}{\rho_0} \right)
\]

whereby \( \tau_z = \tau_s = 0 \):

\[
C_{\infty} = C_{\infty} = \frac{3}{5} \left( 1 - \frac{\tan(L/D)}{L/D} \right) \rho_0^2 \rho^2 L^2 / (D/c)
\]

For small values of the length-to-diameter ratio this may also be written:

\[
C_{\infty} = C_{\infty} = 0, \quad L/D \leq 0.5 : C_{\infty} = C_{\infty} = \frac{3}{5} \left( 1 - \frac{\tan(L/D)}{L/D} \right) \rho_0^2 \rho^2 L^2 / (D/c)
\]

Assuming the lubricant to be oil while the journal material is steel, and setting \( L/D = 0.5 \) and \( c = 10^4 \), it is found that \( C_{\infty} \) and \( C_{\infty} \) equal 6 times the journal mass. This effect could be appreciable for small, compact rotors but should be of no concern in larger machines.

If the complete 2-D deg film is considered to contribute, the results in equations (55) and (56) should be multiplied by 2 such that the factor 3/5 is replaced by 6/5. This latter value may be compared with the results in [11] where the factor equals 1, derived on the basis of an ideal, inviscid fluid. The same result is obtained in [10] from considerations of continuity of accelerated flow.

**Numerical Results**

Calculations are performed for a plain cylindrical journal bearing with film rupture. Three values of the length-to-diameter ratio are considered: \( L/D = 0.1, 0.5 \) and 1.0, and the results are obtained as functions of the static equilibrium eccentricity ratio:

\[
\epsilon = \sqrt{\epsilon_x^2 + \epsilon_y^2}
\]

As the actual values of the Sommerfeld number and the amplitude and velocity coefficients can be found elsewhere, Figs. 1 to 9 only show the corrections to these quantities. Referring to equation (43) the figures give the corrections as:

\[
\text{in Fig. 1: } \Delta \epsilon_x = \Delta \epsilon_x
\]

\[
\text{in Fig. 2: } \Delta \epsilon_x = \Delta \epsilon_x
\]

and analogously for the remaining figures. Hence, to arrive at the true Sommerfeld number and the true coefficients the values obtained on the basis of the conventional Reynolds equation should be multiplied by \( (1 + \lambda \times \text{correction}) \). As the maximum allowable value of \( \lambda \) is of the order of 1 to stay within the linearized regime, the figure shows that the corrections are most amount to a few per cent. When Figs. 2, 3, 8 and 9 give large corrections as tends to zero and Fig. 8 shows a similar simplicity around \( \epsilon \approx 0.7 \) the reason is simply that the base coefficients themselves become zero.

While Figs. 1 to 9 must be multiplied by \( \lambda \) before they are ap-
plotted. Figs. 10 to 12 give the actual acceleration coefficients in the dimensionless form:

\[ CX_X = \frac{C_A}{\rho R (U/c)} \]

and similarly for CYY, CZY, and CXY. Noticing that \( \rho R c \) is the mass of that volume of oil which could be contained in the bearing cavity when the journal is removed, and with \( R/c \) being typically of the order of 10 it is seen that the acceleration coefficients represent an added mass of several times that of the journal itself. For large, heavy rotors the effect is of no importance, but it could become quite significant for small, short rotors. It should be of particular concern in experiments set up to test for the threshold of instability (oil whip) where the journal frequency is a sensible part of the total rotor. If no allowance is made for the virtual mass effect under such conditions serious errors could occur where the experimentally observed critical journal mass would be considerably less than that predicted from a theoretical calculation which does not include the acceleration coefficients.

Conclusions

The analysis and the results confirm that the contribution from the inertial forces to the load carrying capacity and the dynamic reaction forces of journal bearings is quite limited. The corrections to the original amplitude and velocity coefficients amount at most to a few percent which, in general, is less than the tolerance errors and uncertainties from other sources. In practice, therefore, these corrections can be ignored.

In addition to the amplitude and velocity coefficients, there are also associated acceleration coefficients as already shown by Smith [10]. They act as a virtual mass and, for a bearing of typical dimensions and design, the effect may be equivalent to an increase of the journal mass by a factor of as much as 10 to 10. While insignificant in larger machines the effect could be pronounced for small, compact rotors.

Special attention should be given to experimental apparatus for measuring dynamic response or stability of journal bearings where the journal often is a significant part of the total mass for reasons of convenience and simplicity. In such cases, disregarding the acceleration coefficients could lead to serious discrepancies in correlating experimental and theoretical results.

Whereas the analysis should account satisfactorily for the inertial forces in the film flow, the effect that these forces may have on the extent and the instantaneous location of the active film domain has not been explored. Indications are that for sufficiently fast motions of the journal, the film domain will lag behind which probably influences the dynamic characteristics of the bearing for more than the effects examined in the preceding analysis. It is hoped that the presented analysis may serve as a first step in a more comprehensive study of this problem.

References