E. J. GUNTER, JR.
Associate Professor of Mechanical Engineering, University of Virginia, Charlottesville, Va. Mem. ASME

P. R. TRUMPLER
Professor of Mechanical Engineering, University of Pennsylvania, Philadelphia, Pa. Mem. ASME

The Influence of Internal Friction on the Stability of High Speed Rotors With Anisotropic Supports

This paper evaluates the stability of the single mass rotor with internal friction on damped, anisotropic supports. The paper shows under what conditions the rotor stability may be improved by an undamped support with anisotropic stiffness properties. A three dimensional model is presented to show the influence of rotor and support stiffness characteristics on stability. Curves are also presented on how support damping may also improve or even reduce rotor stability. An analog computer solution of the governing equations of motion is presented showing the shaft transient motion for various speed ranges, and also plots of the rotor steady state motion are given for various speeds up to and including the stability threshold. The analysis is used to explain many of the experimental observations of B. L. Newkirk concerning stability due to internal rotor friction.

1 Introduction and Background

Rotor Whirling Due to Internal Friction. During the 1920's, turbine and particularly compressor and pump manufacturers were beginning to construct higher speed, light weight rotors designed for operation above the first critical speed. As more manufacturers adopted "flexible" rotor designs, several encountered severe difficulties when operating above the first critical speed. These problems at first were attributed to the lack of proper balancing. At this time, a major United States manufacturer encountered a series of failures of blast furnace compressors designed to operate above the first critical speed. These machines were subject to occasional fits of more or less violent vibration of unknown origin. During these disturbances the shaft would vibrate at a low frequency which in some cases could be visually observed. The phenomenon was therefore referred to by shop men and engineers as "shaft whipping."

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Dr. B. L. Newkirk of the General Electric Research Laboratory was called in to investigate the nature of the failures. He set up a series of experiments with several units to observe the rotor dynamic behavior. It was observed that at speeds above the first critical speed, these units would enter into a violent whirling in which the rotor centerline precessed at a rate equal to the first critical speed. If the unit rotational speed was increased above its initial whirl speed, the whirl amplitude would increase, leading to eventual rotor failure [1].[1]

During the course of the investigation A. L. Kimball in 1924 suggested that internal shaft friction could be responsible for the shaft whirling [2]. He postulated that below the rotor critical speed the internal friction would damp out the whirl motion while above the critical speed the internal friction would sustain the whirl.

[1] Numbers in brackets designate References at end of paper.

Nomenclature

A = rotor amplification factor = $\frac{\omega_{cro}}{D_2}$, (DIM)
C_1 = foundation damping coefficient, lb-sec/in
C_2 = rotor internal damping coefficient, lb-sec/in
D_1 = foundation damping = $\frac{C_1}{m_2}$; rad/sec
D_2 = rotor internal damping = $\frac{C_2}{m_2}$; rad/sec
D = damping ratio = D_2/D_1
$\mathfrak{D}$ = dissipation function
e_μ = displacement of rotor mass center from shaft elastic centerline, in.
ϕ = polar moment of inertia
K_1 = isotropic foundation stiffness, lb/in
K_2 = rotor stiffness, lb/in
K_x = foundation stiffness in horizontal direction, lb/in
K_y = foundation stiffness in vertical direction, lb/in
L = Lagrangian = $T - V$
m_1 = foundation mass lb-sec²/in
m_2 = rotor mass, lb-sec²/in
qr = generalized coordinate
R = flexibility ratio = K_2/K_y or K_2/K_1 for symmetric support
t = time
T = kinetic energy
T = rotor torque
V = potential energy
X_1 = support horizontal displacement, in.
X_2 = rotor horizontal displacement relative to the support, in.
X = rotor absolute horizontal displacement = $X_1 + X_2$
Y = rotor absolute vertical displacement = $Y_1 + Y_2$, in.
α = foundation flexibility ratio = K_x/K_y (DIM)
δ_m = mass ratio = m_1/m_2, (DIM)
ω = rotor angular velocity, rad/sec
ω_s = rotor stability threshold, rad/sec
ω_{cro} = rotor critical speed on rigid supports = $\sqrt{\frac{K_2}{m_2}}$
ω_{cr} = system critical speed on flexible supports = $\omega_{cro}\sqrt{\frac{1}{1+R}}$
ω_{cx} = rotor critical speed in the horizontal direction, rad/sec
ω_{cy} = rotor critical speed in the vertical direction, rad/sec
θ = angular rotor coordinate
k = K/m_2, equation (21)
ϕ = rotor attitude angle = $\tan^{-1}\frac{X_2}{Y_2}$
$\dot{\phi}$ = rotor precession rate
ρ_2 = $\sqrt{X_2^2 + Y_2^2}$
F_{qr} = generalized force

Later in 1925 Kimball and Lovell [3] performed extensive tests of the internal friction characteristics of various metallic and nonmetallic materials. The experimental technique used to evaluate the magnitude of the internal friction was by measurements of the deflection characteristics of a vertically loaded, horizontal rotating shaft. If the shaft material was perfectly elastic, the application of a vertical load should cause only a corresponding vertical displacement. The presence of material hysteresis in the rotating, deflected, shaft as its segments undergo alternate stress reversal cycles of compression and tension, causes the shaft to deflect sideways. Kimball, by measurement of the shaft vertical inclination angle, was able to determine the ratio of the internal friction forces to the elastic shaft forces. His measurements showed that this ratio for most ferrous and nonferrous materials is between 0.001 and 0.002.

Effect of Shrink Fits on Rotor Internal Friction. Because of the small order of magnitude of the friction forces observed by Kimball, Newkirk concluded that the internal friction created by shrink fits of the impellers and spacers was the predominant cause of the observed whirl instability. He had observed that when all shrink fits were removed from his experimental rotor, no whirl instability could develop. Kimball at Newkirk's suggestion, constructed a special test rotor with rings on hubs shrunk on the shaft [4]. He did indeed confirm Newkirk's conclusion that the frictional effect of shrink fits is a more active cause of shaft whirling than the internal friction within the shaft itself. Measurements showed that, even with the rather light shrinkages used in the tests, the effective internal friction may be increased from two to five times its original value [4]. In fact, Kimball found that long clamping fits always lead to trouble with supercritical speed rotors [5].

For the case of a hub or a sleeve which is fastened to a shaft, which is afterwards deflected, either the surface fibres of the shaft must slip inside the sleeve as they alternately elongate and contract, or the sleeve itself must bend along with the shaft. Usually both actions occur simultaneously to an extent which depends upon the tightness of the shrink fit and the relative stiffness of the two parts. H. D. Taylor, after conducting numerous tests with various hub configurations, concluded that the axial contact length of shrink fits should be as short as permissible and as tight as possible without exceeding the yield strength of the material. Robertson [6] reports that even short, highly stressed shrink fits are not entirely devoid of problems. He states that even small, tight shrink fits may develop whirl instability provided the rotor is given a sufficiently large initial disturbance or displacement to initiate relative internal slippage in the fit. If long shrink fits such as compressor wheels and impeller spacers must be employed, it is important that these pieces be undercut along the central region of the inner bore so that the contact area is restricted to the ends of the shrink fit. Robertson shows several designs of hubs and bosses which have been found to be beneficial in reducing internal friction effects.

Robertson also concludes that a similar effect can be produced by any friction which opposes a change of the deflection of the shaft, such as the friction which exists at the connections of flexible couplings, and even in "rigid" couplings. He referred to this group of friction forces as "hysteretic forces" [7].

Experimental Results and Conclusions of Newkirk on Rotor Whirling. To further investigate all aspects and contributing factors to this problem, Newkirk had an experimental test rotor constructed. Upon extensive testing of this unit, the following important facts were uncovered concerning this phenomenon [1]:

1. The onset speed of whirling or whirl amplitude was unaffected by refinement in rotor balance.
2. Whirling always occurred above the first critical speed, never below it.
3. The whirl threshold speed could vary widely between machines of similar construction.
4. The precession (or whirl) speed was constant regardless of the unit rotational speed.
5. Whirling was encountered only with built-up rotors.
6. Increasing the foundation flexibility would increase the whirl threshold speed.
7. Distortion or misalignment of the bearing housing would increase stability.
8. Introducing damping into the foundation would increase the whirl threshold speed.
9. Increasing the axial thrust bearing load would increase the whirl threshold speed.
10. A small disturbance was sometimes required to initiate the whirl motion in a well balanced rotor.

Of these conclusions, statement 6 concerning the influence of foundation flexibility was the most perplexing to Newkirk. He remained at a loss to explain why foundation flexibility alone should improve the rotor stability. In the early phase of his experimental investigation, his assistant H. D. Taylor discovered that any looseness in the bearing support or clamps which held the test model to the floor, had a strong tendency to prevent whipping. Tests were conducted to determine whether bearing support flexibility alone would prevent whipping or whether additional bearing damping was also required. Conclusions of the experiments indicated that bearing flexibility could prevent rotor whipping even without external bearing damping. Newkirk states in [1], "It is perhaps difficult to accept the view that flexibility only of the bearing support without any attendant damping or energy absorption in the bearing prevents whipping."

Following these experiments, special spring bearings were designed for the unstable turbo-compressors, which incorporated flexibility and damping. Tests were conducted with this bearing arrangement on a three-bearing turbo-compressor rotor using a wide range of stiffness and damping values. In no case could the compressor be made to whip with the flexible bearing support. It was also found that the bearing damping was not necessary to suppress the whip.

It would appear that the introduction of foundation flexibility will lower the rotor first critical speed and hence reduce the whirl threshold speed in the absence of external damping. Instead, in all cases the rotor stability was improved! This question of foundation flexibility became even more of an enigma to Newkirk, when in 1925 he investigated shaft whirling caused by fluid film bearings [8]. When an undamped spring mounted bearing was installed on a rotor which exhibited oil film whirl, it caused a violent whipping to occur instead of suppressing the instability. In this case it was found that friction damping in the spring mounted bearing was essential in stopping the whip motion.

Analysis of the stability of the single mass flexible rotor with internal friction on damped isotropic flexible supports shows that if the foundation flexibility is increased, the rotor stability will be improved only if damping is incorporated into the system [9].

2 General Equations of Motion

Discussion of System. The rotor model employed in [9] is represented by two coupled second order differential equations of motion. This model is inadequate to explain the experimental observations of Newkirk that foundation flexibility alone can improve rotor stability. The derivation of [9] lacks generality and cannot be easily extended to a more complex system which includes bearing mass, large damping forces, anisotropic foundation flexibility and rotor acceleration.

The general equations of motion may be readily obtained from Lagrange's equations of motion provided the system's kinetic, potential, and dissipation functions are known. The only quantity which presents some difficulty is the proper representation of the internal friction forces. The internal friction force cannot be derived from a potential function (system would be inherently stable) but may be obtained from a dissipation function of the proper form. This seems logical since the internal friction assumes the characteristics of a damping force in the subcritical

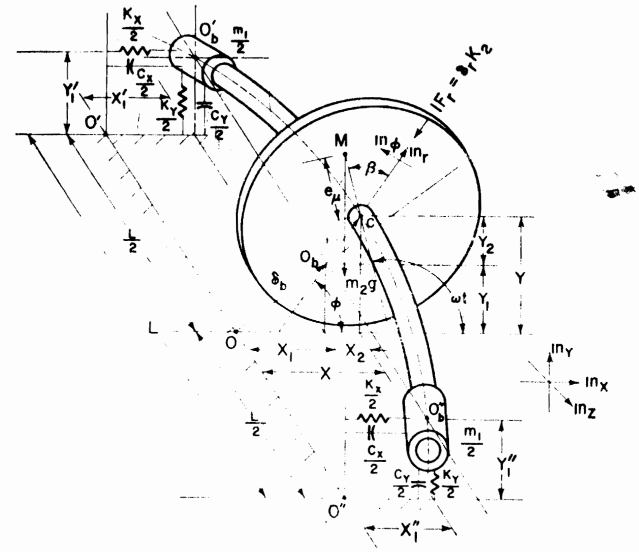

Fig. 1 General single mass rotor on an elastic foundation

speed range.

Thus to properly investigate stability it is necessary to examine the performance of nonconservative systems in which the dissipation function has particular properties. For example [10] shows that the conservative Jeffcott system derived from only the potential and kinetic energy does not produce self-excited whirl, contrary to the papers by Kane [11] and Soderberg [12].

The system to be analyzed is shown in Fig. 1, which represents the extended Jeffcott rotor to include foundation flexibility and bearing mass. The mass of the rotor is contained in a plane normal to line $O'O''$ which is located midway between the two bearing locations. The major assumptions which will be employed in this analysis are:

1 No gyroscopic forces.

2 The characteristics and displacements at both bearing housings are identical.

3 The relative shaft-bearing displacements are negligible in comparison to the absolute rotor and support displacements.

4 The rotor total angular velocity is constant.

5 No gravitational forces.

The above assumptions will reduce the system to four coupled second order equations of motion.

Derivation of General Equations of Motion. The position vectors to the mass stations are given by

m_1: bearing mass

$$\overset{0 \rightarrow M_1}{P} = X_1 \vec{n}_x + Y_1 \vec{n}_y \tag{1}$$

m_2: rotor mass

$$\overset{0 \rightarrow M_2}{P} = (X_1 + X_2 + e_\mu \cos \omega t)\vec{n}_x + (Y_1 + Y_2 + e_\mu \sin \omega t)\vec{n}_y \tag{2}$$

If ωt is replaced by θ, the system possesses five degrees of freedom and hence five equations of motion will be required to completely describe the system.

The velocities of the mass stations are given by

m_1:

$$\overset{R \rightarrow M_1/0}{V} = \dot{X}_1 \vec{n}_x + \dot{Y}_1 \vec{n}_y \tag{3}$$

m_2:

$$\overset{R \rightarrow M_2/0}{V} = (\dot{X}_1 + \dot{X}_2 - e_\mu \dot{\theta} \sin \theta)\vec{n}_x + (\dot{Y}_1 + \dot{Y}_2 + e_\mu \dot{\theta} \cos \theta)\vec{n}_y \tag{4}$$

The kinetic energy of the system is given by:

$$T = \frac{1}{2}\{m_2[(\dot{X}_1 + \dot{X}_2 - e_\mu \dot{\theta} \sin \theta)^2 + (\dot{Y}_1 + \dot{Y}_2 + e_\mu \dot{\theta} \cos \theta)^2] + m_1[\dot{X}_1^2 + \dot{Y}_1^2] + \Phi \dot{\theta}^2\} \tag{5}$$

The potential energy of the system is given by:

$$V = \frac{1}{2}[K_x X_1^2 + K_y Y_1^2] + \frac{1}{2} K_2 [X_2^2 + Y_2^2] \tag{6}$$

The dissipation function caused by the external damping is given by:

$$\mathfrak{D}_1 = \frac{1}{2} C_1 [\dot{X}_1^2 + \dot{Y}_1^2] \tag{7}$$

and the dissipation function caused by the internal rotor friction can be obtained from the force relationship equation (9) and is given by [9]:

$$\mathfrak{D}_2 = C_2 \left[\frac{\dot{X}_2^2 + \dot{Y}_2^2}{2} + \omega (Y_2 \dot{X}_2 - X_2 \dot{Y}_2) \right] \tag{8}$$

Close investigation of the dissipation function $\mathfrak{D}_2$ shows that it may be expressed in the following form:

$$\mathfrak{D}_2 = C_2 \left[\frac{1}{2} \left(\overset{RC/0_b}{V} \right)^2 + \omega \rho_2^2 \frac{d\phi}{dt} \right] \tag{9}$$

Thus when the rotor precession rate $\dot{\phi}$ is zero, the internal friction dissipation function assumes the characteristics of conventional viscous damping. It is very important to note that only in the case when the dissipation function has this special characteristic of being dependent upon the rotor or bearing precession rate can self-excited whirl instability be developed. When the system damping terms are represented entirely by functions of the form in equation (7) the system is inherently stable.

The governing equations of motion of the system are obtained from Lagranges equation which states:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} + \frac{\partial \mathfrak{D}}{\partial \dot{q}_r} = F_{qr} \tag{10}$$

where $L = T - V$.

Application of the above for the five generalized coordinates yields the following equations:

$$X_1: \quad m_1 \ddot{X}_1 + m_2[\ddot{X}_1 + \ddot{X}_2 - e_\mu \ddot{\theta} \sin \theta + e_\mu (\dot{\theta})^2 \cos \theta] + C_1 \dot{X}_1 + K_x X_1 = 0 \tag{11}$$

$$Y_1: \quad m_1 \ddot{Y}_1 + m_2[\ddot{Y}_1 + \ddot{Y}_2 + e_\mu \ddot{\theta} \cos \theta + - e_\mu (\dot{\theta})^2 \sin \theta] + C_1 \dot{Y}_1 + K_y Y_1 = 0 \tag{12}$$

$$X_2: \quad m_2[\ddot{X}_1 + \ddot{X}_2 - e_\mu \ddot{\theta} \sin \theta - e_\mu (\dot{\theta})^2 \cos \theta] + C_2[\dot{X}_2 + \omega Y_2] + K_2 X_2 = 0 \tag{13}$$

$$Y_2: \quad m_2[\ddot{Y}_1 + \ddot{Y}_2 + e_\mu \ddot{\theta} \cos \theta - e_\mu (\dot{\theta})^2 \sin \theta] + C_2(\dot{Y}_2 - \omega X_2) + K_2 Y_2 = 0 \tag{14}$$

$$\theta: \quad \Phi \ddot{\theta} + m_2[-(\ddot{X}_1 + \ddot{X}_2) e_\mu \sin \theta + (\ddot{Y}_1 + \ddot{Y}_2) e_\mu \cos \theta + e_\mu^2 \ddot{\theta}] = T \tag{15}$$

Neglecting rotor acceleration, the five governing equations reduce to the four following equations:

$$(1 + \delta m)\ddot{X}_1 + \ddot{X}_2 + D_1 \dot{X}_1 + \omega_x^2 X_1 = e_\mu \omega^2 \cos \omega t \tag{16}$$

$$(1 + \delta m)\ddot{Y}_1 + \ddot{Y}_2 + D_1 \dot{Y}_1 + \omega_y^2 Y_1 = e_\mu \omega^2 \sin \omega t \tag{17}$$

$$\ddot{X}_1 + \ddot{X}_2 + D_2 \dot{X}_2 + D_2 \omega Y_2 + \omega_2^2 X_2 = e_\mu \omega^2 \cos \omega t \tag{18}$$

$$\ddot{Y}_1 + \ddot{Y}_2 + D_2 \dot{Y}_2 - D_2 \omega X_2 + \omega_2^2 Y_2 = e_\mu \omega^2 \sin \omega t \tag{19}$$

where

$$\delta m = \frac{m_1}{m_2} \qquad \omega_y{}^2 = \frac{K_y}{m_2}$$

$$\omega_x{}^2 = \frac{K_x}{m_2} \qquad \omega_2{}^2 = \frac{K_2}{m_2} = \omega_{cro}{}^2$$

3 General Stability Analysis

Routh Criterion. A very general stability analysis of the rotor system as shown in Fig. 1 may now be made. This analysis includes the effects of bearing mass and is not restricted to small values of damping as is the analysis of reference [9].

Assume a complementary solution to equations (16–19) of the form:

$$X_1 = Ae^{\lambda t}; \qquad Y_1 = Be^{\lambda t}$$

$$X_2 = Ce^{\lambda t}; \qquad Y_2 = De^{\lambda t}$$

results in:

$$\begin{bmatrix} [\lambda^2(1+\delta m) + D_1\lambda + k_x] & 0 & \lambda^2 & 0 \\ 0 & [\lambda^2(1+\delta m) + D_1\lambda + k_y] & 0 & \lambda^2 \\ \lambda^2 & 0 & [\lambda^2 + \lambda D_2 + k_2] & D_2\omega \\ 0 & \lambda^2 & -D_2\omega & [\lambda^2 + D_2\lambda + k_2] \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0 \qquad (20)$$

Since the coefficients A, B, C, D are arbitrary, the determinant of the coefficients must vanish. Expanding the determinant we obtain the following eight-order frequency equation in λ.

$$\begin{aligned} &\{\delta m\}^2\lambda^8 + \{2\delta m[(1+\delta m)D_2 + D_1]\}\lambda^7 \\ &\quad + \{(1+\delta m)^2(2k_2 + D_2{}^2) + (3 + 4\delta m)D_1D_2 \\ &\quad + \delta m(k_y + k_x) - 2(1+\delta m)k_2 + D_1{}^2\}\lambda^6 \\ &\quad \times \{2(1+\delta m)[D_1(2k_2 + D_2{}^2) + D_2k_2(1+\delta m)] \\ &\quad + D_1(k_x + k_y - 2k_2) + D_2[(1 + 2\delta m)(k_x + k_y) \\ &\quad + 2D_1{}^2]\}\lambda^5 + \{(1+\delta m)[(k_x + k_y)(2k_2 + D_2{}^2) \\ &\quad + 4D_1D_2k_2 + (1+\delta m)(D_2{}^2\omega^2 + k_2{}^2)] + 2D_1D_2(k_x + k_y) \\ &\quad + 2D_1{}^2k_2 + D_1{}^2D_2{}^2 + k_xk_y - k_2(k_x + k_y)\}\lambda^4 \\ &\quad + \{(1+\delta m)[k_2(k_x + k_y)2D_2 + 2D_1(D_2{}^2\omega^2 + k_2{}^2)] \\ &\quad + D_1(k_x + k_y)(2k_2 + D_2{}^2) + 2D_2(D_1{}^2k_2 + k_xk_y)\}\lambda^3 \\ &\quad + \{(1+\delta m)(k_x + k_y)(D_2{}^2\omega^2 + k_2{}^2) + 2D_1D_2k_2(k_x + k_y) \\ &\quad + (D_2{}^2\omega^2 + k_xk_y) + k_2(D_1{}^2k_2 + 2k_xk_y)\}\lambda^2 \\ &\quad + \{D_1(D_2{}^2\omega^2 + k_2{}^2)(k_x + k_y) + 2D_2k_2k_xk_y\}\lambda \\ &\quad + k_xk_y[D_2{}^2\omega^2 + k_2{}^2] = 0 \qquad (21) \end{aligned}$$

equation (21) is of the form:

$$\sum_{K=0}^{N=8} A_{n-k}\lambda^k = 0$$

Thus instead of a fourth order equation as was obtained in [9], the introduction of bearing mass and large damping forces requires an eight-order system to be evaluated. For systems larger than fourth-order, the Routh-Hurwitz determinant method becomes cumbersome and unwieldy to use. In such cases, the original Routh method is preferable [13]. This method is outlined as follows:

Consider the following array of coefficients:

$$\begin{array}{ccccc} A_0 & A_2 & A_4 & A_6 & A_8 \\ A_1 & A_3 & A_5 & A_7 & \\ C_1 & C_2 & C_3 & C_4 & \\ D_1 & D_2 & D_3 & & \\ E_1 & E_2 & E_3 & & \\ F_1 & F_2 & & & \\ G_1 & G_2 & & & \\ H_1 & & & & \end{array}$$

where

$$\begin{aligned} C_1 &= A_2 - A_0A_3/A_1; \quad C_3 = A_6 - A_0A_7/A_1 \\ C_2 &= A_4 - A_0A_5/A_1; \quad C_4 = A_8 \\ D_1 &= A_3 - A_1C_2/C_1; \quad D_3 = A_7 - A_1C_4/C_1 \\ D_2 &= A_5 - A_1C_3/C_1 \\ E_1 &= C_2 - C_1D_2/D_1; \quad E_3 = D_3 \\ E_2 &= C_3 - C_1D_3/D_1 \\ F_1 &= D_2 - D_1E_2/E_1; \quad F_2 = D_3 - D_1E_3/E_1 \\ G_1 &= E_2 - E_1F_2/F_1; \quad G_2 = E_3 \\ H_1 &= F_2 - F_1G_2/G_1 \end{aligned}$$

The necessary and sufficient condition of stability is that all of the coefficients of the first column of the array must be positive.

Stability for Light Damping. If the rotor internal and external damping forces are considered as small in comparison to the shaft elastic restoring forces and the bearing housing mass is neglected, then one of the displacement variables may be eliminated to obtain [9] a single fourth order equation in either x or y to represent the system. For example, elimination of the y coordinate results in the following fourth order equation in x:

$$\begin{aligned} &\ddot{\ddot{X}} + [C_x + C_y]\dddot{X} + [\omega_{cx}{}^2 + \omega_{cy}{}^2 + C_xC_y]\ddot{X} \\ &\quad + [\omega_{cx}{}^2C_y + \omega_{cy}{}^2C_x]\dot{X} + [\omega_{cx}{}^2\omega_{cy}{}^2 + \omega^2\mu_x\mu_y]X = 0 \qquad (22) \end{aligned}$$

$$C_x = \mu_x + \nu_x$$

$$C_y = \mu_y + \nu_y$$

where

$$\mu_x = \frac{C_2}{m_2}\left(\frac{K_y}{K_x + K_2}\right)^2 = D_2\left(\frac{1}{\alpha + R}\right)^2$$

$$\mu_y = \frac{C_2}{m_2}\left(\frac{K_y}{K_y + K_2}\right)^2 = D_2\left(\frac{1}{1 + R}\right)^2$$

$$\nu_x = \frac{C_1}{m_2}\left(\frac{K_2}{K_x + K_2}\right)^2 = D_1\left(\frac{R}{\alpha + R}\right)^2$$

$$\nu_y = \frac{C_1}{m_2}\left(\frac{K_2}{K_y + K_2}\right)^2 = D_1\left(\frac{R}{1 + R}\right)^2$$

$$D_1 = \frac{C_1}{m_2}, \; D_2 = \frac{C_2}{M_2}, \; R = \frac{K_2}{K_y}, \; \alpha = \frac{K_x}{K_y}$$

ω_{cx} = natural system resonance frequency for the X direction.

$$= \sqrt{\frac{K_2K_x}{m_2K_2 + K_x}} = \omega_{cro}\sqrt{\frac{\alpha}{R + \alpha}}$$

ω_{cy} = natural system resonance frequency for the Y direction

$$= \sqrt{\frac{K_2K_y}{m_2(K_2 + K_y)}} = \omega_{cro}\sqrt{\frac{1}{1 + R}}$$

ω_{cro} = rotor natural resonance frequency on rigid supports.

By the application of the Routh-Hurwitz stability criterion for $N = 4$ the rotor threshold of stability is obtained after some algebraic manipulation as follows:

$$\omega_s = \omega_{cro}\sqrt{F_1 + F_2} \qquad (23)$$

where

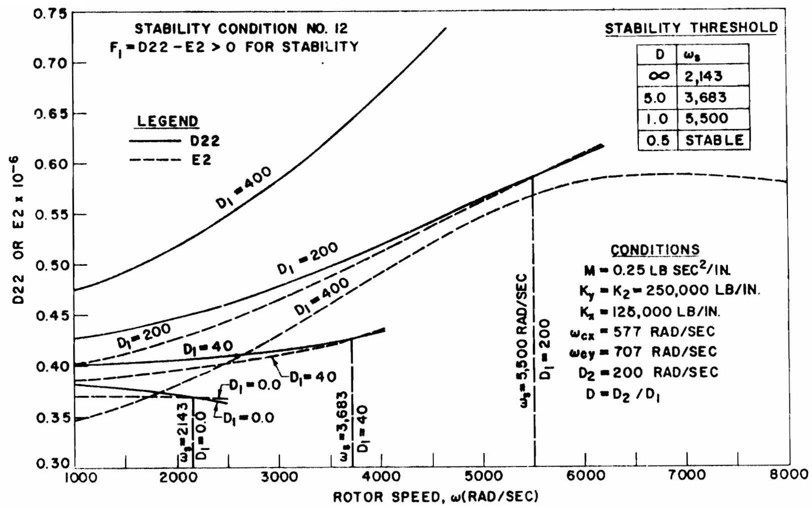

Fig. 2 Routh stability coefficients at various rotor speeds

$$F_1 = \left(\frac{\omega_{cro}}{D_2}\right)^2 \frac{[R^2 + D][R^2 + D\alpha^2]R^2(\alpha + R)^2(1 + R)^2(1 - \alpha)^2}{\{\alpha[(1 + R)^2(R^2 + D\alpha^2) + (\alpha + R)^2(R^2 + D)]\}^2}$$

$$F_2 = \frac{(R^2 + D)(R^2 + D\alpha^2)}{D^2\alpha^2} \times \left[\frac{\alpha(R + \alpha)(R^2 + D) + (R + 1)(R^2 + D\alpha^2)}{(R + 1)^2(R^2 + D\alpha^2) + (R + \alpha)^2(R^2 + D)}\right]$$

Digital Computer Program. A digital computer program was developed to calculate the threshold of stability of equation (21) by the general Routh procedure as outlined. Since the coefficients of the characteristic equation are speed dependent, an iterative approach was employed to obtain the threshold speed. The method consists of assuming an initial value of ω and calculating the coefficients A_0 to A_8. If these are all positive, the program continues, calculates the Routh coefficients and tests to see if any coefficient in the first column is negative. If no negative coefficient appears, the initial value of ω is incremented to a new value and the process is repeated. If a negative coefficient appears, the next value of ω is obtained by averaging the unstable speed with the last previous stable value. This procedure is continued until a convergence criterion is satisfied.

In the majority of cases examined, the Routh coefficient F_1 was found to be the term which indicates the system stability. In the computer program F_1 is determined by the difference between $D22$ and $E2$, Fig. 2 represents a plot of these two functions—for various values of external damping and over a range of rotor speeds. The threshold of stability is determined by the intersection of $D22$ and $E2$. In a number of cases, the values of $D22$ and $E2$ are only slightly different. As an example, for $D_1 = 200$ rad/sec, the values of $D22$ and $E2$ are very close and it is difficult to determine the intersection point from the inspection of the plot of the two functions. As the value of the external damping increases, a point is reached in which the two functions will no longer intersect. Note, in Fig. 2, for $D_1 = 400$ rad/sec, $D22$ and $E2$ rapidly diverge. In this case, the fact that the coefficients G_1 and H_1 are also positive, seems to indicate that the system is stable.

Since the numerical operations involve the sum and differences of some large numbers, it was considered possible that numerical instability could be occurring, particularly for such cases as $D_1 = 400$ of Fig. 2 which shows the two functions $D22$ and $E2$ diverging. To insure greater accuracy in the calculations, double precision was used and all calculations were carried out to 16 place accuracy.

Table 1 represents a comparison between some typical values of the approximate system of equation (23) and the general solution of equation (21). When bearing mass is neglected equation (21) reduces to a 6th order system. The table shows that for zero foundation damping ($D_1 = 0$), the approximate solution indicates a slightly higher threshold of stability. When external damping is introduced into the system, the stability threshold for the exact solution increases rapidly. At $D = 1$ ($D_1 = 200$ rad/sec) the general computer solution indicates that the threshold value is over 40 percent larger than the approximate analysis. When the external damping was doubled to $D_1 = 400$ rad/sec the computer program did not find any negative values in the leading Routh coefficients. (When the rotor speed ω reaches 100 times the critical speed, the system is assumed stable and the iterative procedure is discontinued.)

Table 1 Comparison of dimensionless approximate and exact stability threshold speeds for various values of supporting damping

δM = 0.0 R = 1.0 A = 5.0 α = 0.5 D_2 = 400				
	D_1 = 0	D_1 = 40	D_1 = 200	D_1 = 400
Approx. 4th Order	2.39	2.62	3.23	4.21
Exact	2.143	3.683	5.50	Stable
% Deviation	+11.5%	-29%	-41%	--

Rotor Stability With Symmetric Bearing Support. Fig. 3 represents the stability characteristics of a rotor on a symmetric foundation with conditions that correspond exactly to Fig. 4 of [9]. Comparison of the two charts shows that for zero foundation damping, the approximate and the exact solutions coincide. Both predict the rotor will be unstable above the critical speed. As external damping is introduced into the system, the rotor stability increases in much the same manner as given by the approximate solution. In fact, considering the number of terms deleted in the analysis, it provides a surprisingly accurate representation of the rotor stability characteristics.

The important point which the general analysis brings out, which is not obtained from the approximate solution, is that for each given value of external damping there is a value of foundation flexibility which will make the system entirely stable for all speeds. For example, in Fig. 3 for $D = 0.2$ the rotor is completely stable for any flexibility ratio $R > 1.0$. As the foundation damping is decreased, greater foundation flexibility is required to completely stabilize the rotor.

To verify that this is indeed the case, the governing equations of motion (equations 16–19) were programmed on the analog computer. The analog program was run for conditions corresponding to $D = 0.2$, $R = 1.0$. It was found that when the R

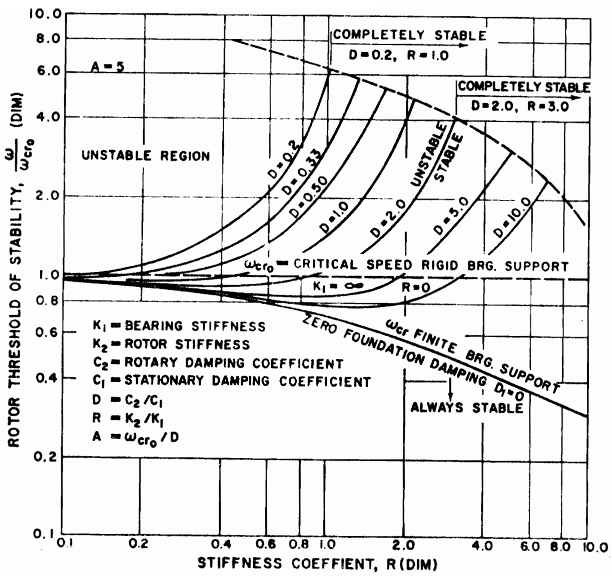

Fig. 3 Effect of external damping and bearing flexibility on the rotor whirl threshold speed-symmetric support

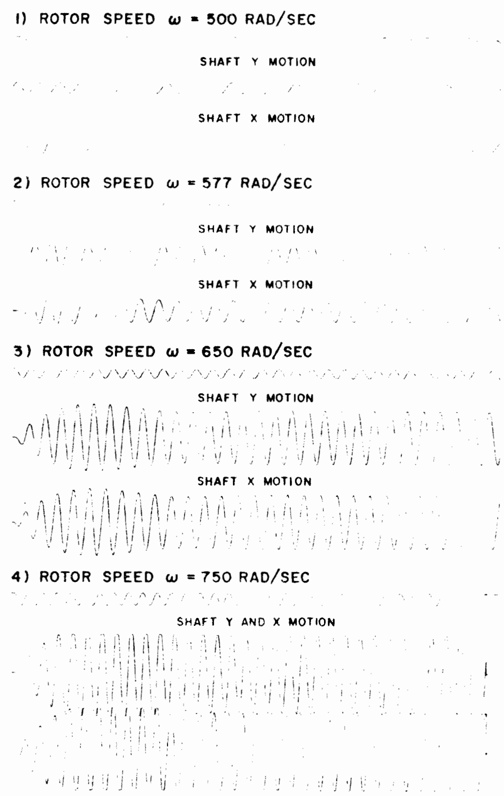

Fig. 4 Transient motion of an unbalanced rotor with internal friction damping, ω = 500 − 750 rad/sec

value was increased above 1.0, no rotor instability was observed.

Example Problem 1

Let:

m_2 = 0.25 lb-sec²/in (96.6 lb rotor)

K_2 = shaft stiffness = 250,000 lb/in

K_1 = support stiffness = 250,000 lb/in, R = 1

$D_1 = D_2$ = 200 rad/sec; D = 1

$\omega_{cro} = \sqrt{\frac{k_2}{m_2}}$ = 1000 rad/sec = rotor natural frequency (on rigid supports)

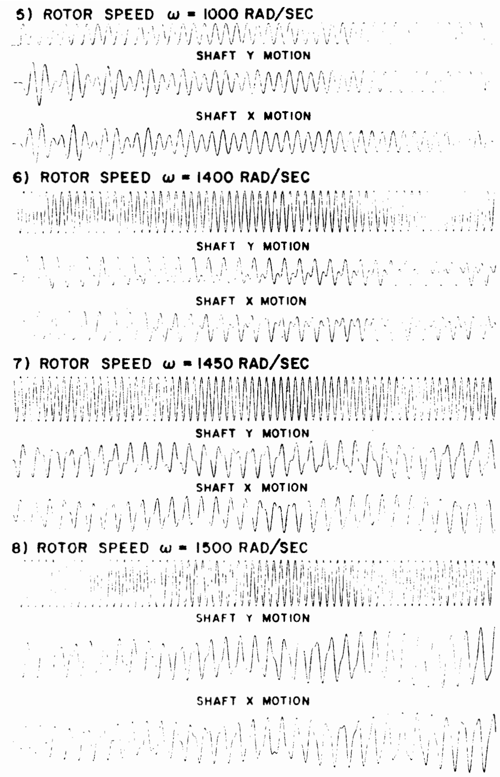

Fig. 5 Transient motion of an unbalanced rotor with internal friction damping, ω = 1,000 − 1,500 rad/sec

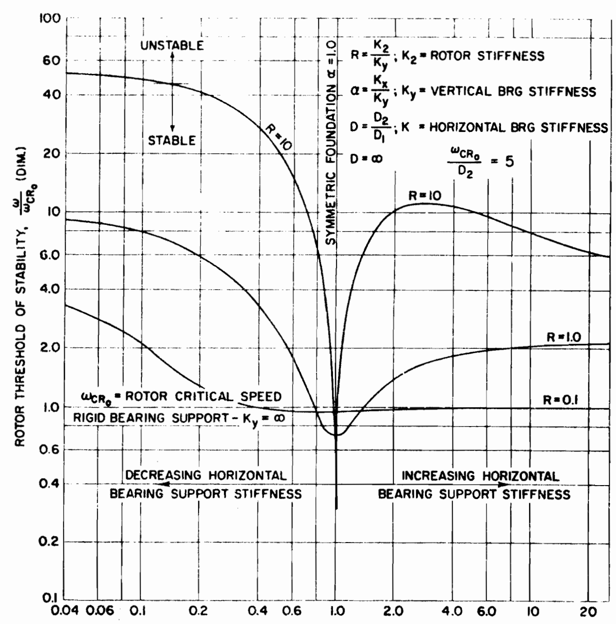

Fig. 6 Effect of unsymmetric bearing support flexibility on rotor whirl threshold speed-zero foundation damping

ω_{cr} = 706 rad/sec = system natural frequency

A = critical speed amplification factor

$$= \frac{\omega_{cro}}{D} = 5$$

The rotor stability threshold speed on rigid supports is 1000 rad/sec. Fig. 3 shows that for D = 1 and R = 1, the stability threshold is raised to 1414 rad/sec. If the support damping D_1

were zero, then the stability threshold would be reduced.

A number of rotor orbits and shaft x-y traces were obtained of an unbalanced rotor for conditions identical to example 1. The effect of the unbalance, as shown in Fig. 4, is to produce a steady-state rotor displacement similar to Fig. 7 of reference [9]. When the rotor is released, the unbalance causes the rotor to spiral out and then settle down to a stable synchronous motion when the rotor speed is below the threshold speed. As the rotor speed increases, the size of the synchronous rotor motion increases and reaches a maximum at the rotor critical speed of $\omega_{cr} = 706$ rad/sec.

In Fig. 5, which illustrates the rotor motion in the speed range of 1000 to 1500, a sizeable transient motion begins to develop. In run 6, a nonsynchronous transient motion is superimposed upon the rotor synchronous precession. After the transient motion is suppressed, the motion is stable synchronous precession as caused by unbalance. Notice that since we are above the rotor critical speed the shaft x-y displacements are considerably smaller than those shown in Fig. 4. As the speed is increased, the nonsynchronous component becomes more predominant until it completely overshadows the synchronous component as shown by run 8 for $\omega = 1500$ rad/sec.

Rotor Stability With Unsymmetric Bearing Support. Fig. 6 shows as a comparison the stability for stiffness values of $R = 0.1$, 1.0, and 10.0 with zero foundation damping. Examination of the stability curve for $R = 10$ shows clearly the influence of even small changes in α on stability. For example, for $R = 10$ and $\alpha = 1$, the threshold is 0.3 of the rotor critical speed. Increasing α to 2 improves the threshold by a factor of 10 while a reduction in α to 0.5 causes the stability threshold ratio to increase to 20.

At low values of R, that is when the vertical foundation is considerably stiffer than the rotor, very little change in performance is obtained by varying the horizontal stiffness. As the values of R increase, a change of α from unity causes an increase in stability. Notice that stability is improved both by increasing as well as decreasing the horizontal bearing stiffness. There are asymptotes to the stability limit that will be obtained as α approaches zero or infinity. Note that for rotor stiffness ratios of $R = 1$ or higher and $\alpha > 1$, there are optimum values of α for each R value to obtain maximum stability. Increasing the horizontal stiffness above this value causes a reduction in stability. Thus little instability is gained by having α greater than 3.

To more vividly illustrate the rotor stability characteristics in the absence of foundation damping, a three-dimensional stability model was developed as shown in Fig. 7. Fig. 7 shows the stability model viewed in the direction of increasing R. The model profile is fairly level for low values of R. As the vertical foundation flexibility increases, the rotor critical speed diminishes, as represented by the center line $\alpha = 1.0$.

Fig. 8 represents the rotor stability characteristics for $R = 1$ and $A = 5$, for various values of support damping. With no foundation damping present, the exact and approximate solutions are almost identical for $\alpha > 1$. When $\alpha < 1$, the approximate solution indicates a higher threshold of stability. In both cases, reduction of horizontal bearing flexibility and increase in foundation damping produce a rapid rise in the stability threshold.

In all cases, the approximate solution predicts that increasing foundation damping will always raise the rotor threshold speed for a given value of α and R. The exact solution shows a very interesting phenomena that for large values of α, increasing the foundation damping may actually reduce the stability threshold. In Fig. 8, the value of $\alpha = 3$ represents a cross-over point with respect to the influence of damping. For example, at $R = 3$, and $D = 1.0$, the dimensionless threshold speed is 2.319. Increasing the external damping by a factor of 5 ($D = 0.20$) only causes the threshold speed to increase to 2.381.

For values of $\alpha < 3$, the higher value of external damping actually produces a lower threshold speed. If the foundation is constrained in the horizontal direction, and only vertical motion permitted, the value of $\alpha \to \infty$. In this case we find that light

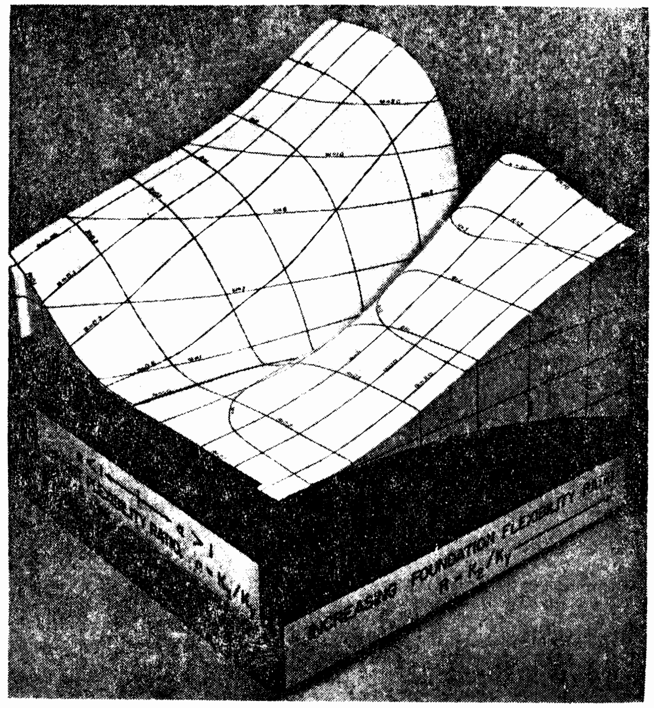

Fig. 7 Topological model of rotor stability characteristics with zero foundation damping

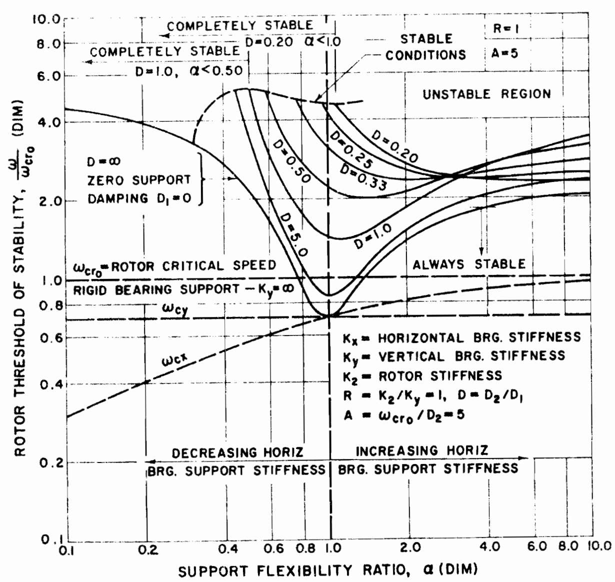

Fig. 8 Effect of unsymmetric bearing support flexibility on the rotor whirl threshold speed for R = 1.0 and A = 5.0 — general system

values of external damping will improve stability but there exists a limiting damping value which will result in a reduction of the threshold. For the case of extremely high external damping, the threshold speed is depressed down to the rotor critical speed (the foundation is acting as if it were rigid).

Example Problem 2

Let

$$m_2 = 0.25 \text{ lb-sec}^2\text{/in (96.6 lb rotor)}, \; m_1 = 0$$
$$k_2 = k_y = 250{,}000 \text{ lb/in}$$
$$k_x = 125{,}000$$
$$D_1 = D_2 = 200 \text{ rad/sec}$$
$$\omega_{cx} = 577 \text{ rad/sec} \qquad \omega_{cy} = 707 \text{ rad/sec}$$

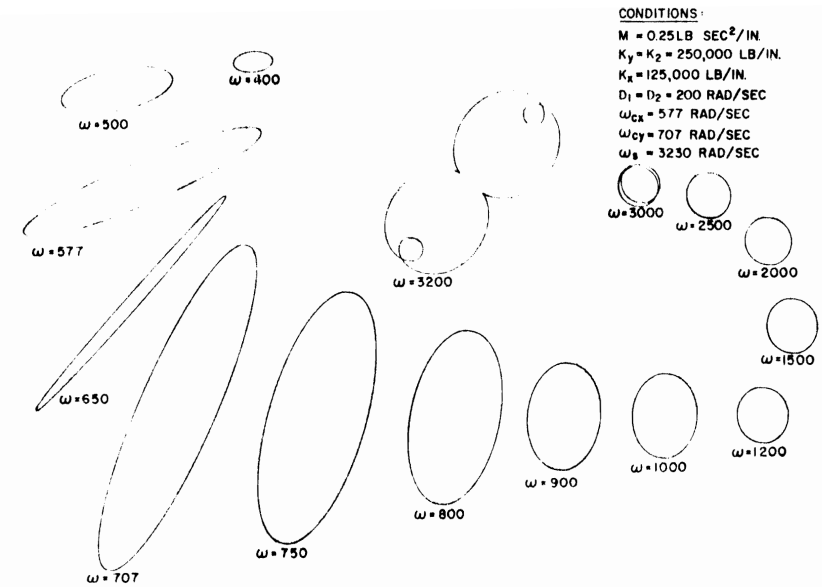

Fig. 9 Steady-state whirl orbits of an unbalanced rotor with foundation asymmetry ω = 400 to 3,200 rad/sec

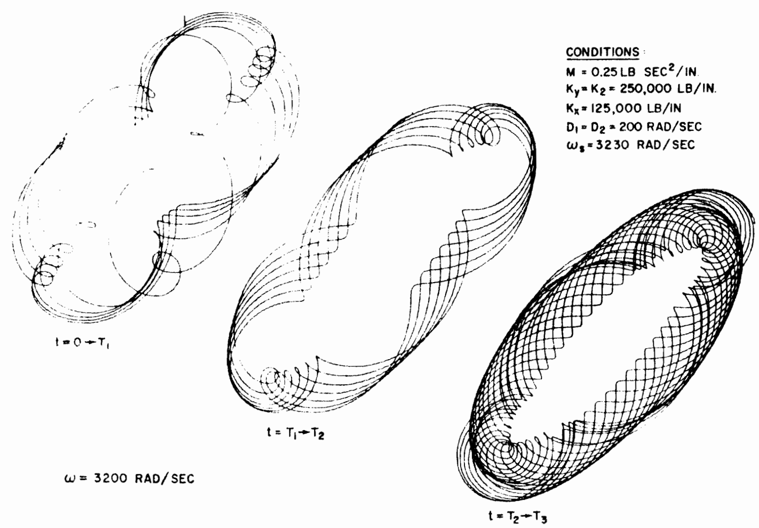

Fig. 10 Whirl orbit of an unbalanced rotor at the threshold of stability $-\omega = \omega_s$

Example 2 is identical to 1 except the support stiffness in the horizontal direction has been reduced in half. Fig. 9 represents the steady state whirl orbit of an unbalanced rotor obtained from the analog computer. Reduction of the horizontal stiffness increases the rotor stability threshold from 1414 to over 3200 rad/sec. Fig. 10 represents the rotor whirl orbit at the stability threshold.

Effect of Magnitude of Internal Damping on Stability. Fig. 11 represents the rotor stability characteristics for a range of values of internal damping and no external damping. The smaller the value of A, the larger the amount of internal friction damping. For example, if we assume the rotor critical speed to be 1,000 rad/sec, the value of $A = 10$ would correspond to a damping value of $D_2 = 100$ rad/sec and $A = 1$ is equivalent to $D_2 = 1{,}000$ rad/sec. Actual measurements of the internal damping factor by Kimball indicates that A should be of the order of 5 to 10 or larger. In this case Fig. 11 shows that large increases in rotor stability are possible by the introduction of bearing asymmetry for $A > 5$. Fig. 11 shows the important conclusion that the larger the internal friction becomes, the less the effectiveness of bearing asymmetry on improving the stability. In fact, for the value of $A = 1$, the rotor stability threshold is below the original rotor critical speed. In this case rotor stability may be improved only by the addition of external damping.

Fig. 11 is important in another respect, as it indirectly answers the question posed by Dr. Newkirk in 1925. That is why foundation flexibility will improve rotor stability in the case of fluid film bearings. The fluid film bearing produces a cross-coupled force relationship similar to internal friction [10]. In the case of a fluid film bearing, the force component which is responsible for the rotor instability is not necessarily small, as is the case with the rotor internal friction, but can be of the same order as the rotor elastic forces. Thus the ratio A may be of the order unity for a fluid film bearing. In this case the introduction of foundation flexibility will result in a reduction of the rotor threshold speed and only if external foundation damping is added can the threshold be raised.

4 Summary and Conclusions

The introduction of internal rotor friction will cause unstable, nonsynchronous rotor precession above the critical speed. The analysis of the precession rate shows that the nonsynchronous

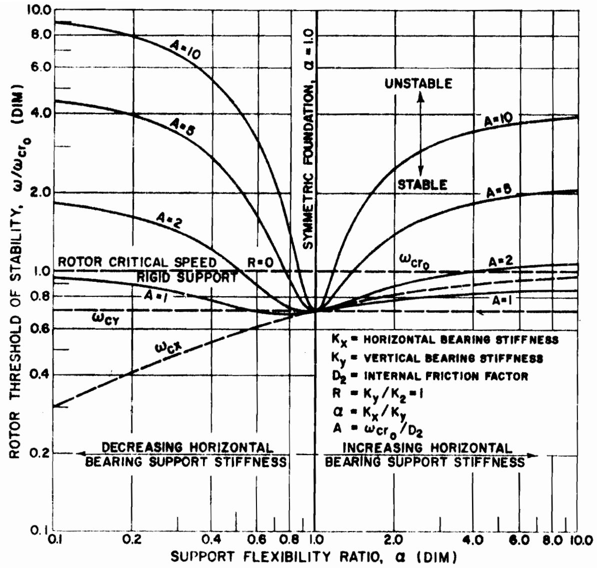

Fig. 11 Effect of unsymmetric bearing support on the rotor whirl threshold speed for various values of internal friction—zero foundation damping

motion is approximately equal to the rotor critical speed and remains constant over a considerable speed range. This is in accordance with the observations of Newkirk [1] and Kushul [14] on the rotor whirl.

One of the most important aspects of this paper is the influence of foundation flexibility and damping on the rotor stability. A symmetric flexible foundation will reduce the rotor critical speed and also the whirl threshold in the absence of external damping. If external damping is added, the stability threshold can be greatly improved. The interesting aspect of the problem is that foundation asymmetry alone, without foundation damping, can create a large increase in the whirl threshold speed.

The stability analysis of the general equations of motion (21) reveals a number of important stability characteristics which are not obtained from the analysis of equation (23) of reference [9] assuming light damping. The general analysis shows that under certain conditions the rotor can be completely stabilized by foundation flexibility and damping. The approximate analysis indicates that the greater the external foundation damping, the higher the threshold of stability will be. The general analysis shows that there is a limiting value of external damping that should be used, and that values higher than this will result in a reduction of stability.

The analysis of the effect of bearing mass on stability shows that bearing mass will lower the threshold for the symmetric case. One interesting aspect of bearing mass is that the mass of the bearing and flexibility of the support may be designed to attenuate the displacement amplitude at the rotor critical speed; that is to act as a dynamic vibration absorber. This seems to have a deleterious effect on the rotor stability.

Acknowledgments

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