Robert E. Cunningham
Mem. ASME

David P. Fleming
Mem. ASME

NASA Lewis Research Center,
Cleveland, Ohio

Edgar J. Gunter
University of Virginia,
Charlottesville, Va.
Mem. ASME

Design of a Squeeze-Film Damper for a Multi-Mass Flexible Rotor

A single mass flexible rotor analysis was used to optimize the stiffness and damping of a flexible support for a symmetric five-mass rotor. The flexible support attenuates the rotor motions and forces transmitted to the support bearings when the rotor operates through and above its first bending critical speed. An oil squeeze-film damper was designed based on short bearing lubrication theory. The damper design was verified by an unbalance response computer program. Rotor amplitudes were reduced by a factor of 16 and loads reduced by a factor of 36 compared with the same rotor on rigid bearing supports.

Introduction

In rotating machinery such as turbojet engines, compressors, and turbines, the rotors often encounter high vibrational amplitudes resulting in large forces transmitted to the bearings and the support structure. The problem becomes more severe as machinery is designed to be lighter and, hence, more flexible. High vibrational responses may be due to several causes and may be roughly grouped under the headings of synchronous and nonsynchronous response.

Synchronous response is usually associated with unbalance in the rotor. This unbalance may result from either the manufacturing process or the assembly of the components. Even if a rotor is well balanced initially, the balance degrades with rotor use. Thermal gradients can cause warping of the shaft. Erosion of compressor or turbine blades can alter the balance of the rotor. Therefore in the design of machinery provisions should be made so that the increase of unbalance with operation will not overload the bearings or cause excessive rotor amplitudes.

Another serious problem of high-speed turbomachinery is the occurrence of nonsynchronous, self-excited whirl motion. This is commonly associated with fluid-film bearings, but can also be caused by rotor internal friction [1][1] or variable aerodynamic loading [2].

Theoretical studies indicate that problems of both self-excited

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[1] Numbers in brackets designate References at end of paper.

rotor instability and high unbalance (synchronous) response can often be greatly alleviated by a properly designed damping system at the rotor supports (e.g., references [1, 3]). Flexible damped rotor supports may be used to

(a) Reduce the forces transmitted through the bearings and foundation

(b) Reduce the amplitudes of motion of the rotor which could result in rubbing and excessive wear of close fitting components

(c) Permit smooth operation through critical speeds

(d) Protect the machine from sudden buildup of unbalance forces due to compressor or turbine blade loss

(e) Protect the machine from potentially destructive, self-excited instability

This report will consider steady, synchronous motion; that is, effects (a), (b), and (c).

Damping may be achieved by various mechanisms, such as Coulomb friction, viscoelastic materials, and viscous dampers which may utilize either compressible or incompressible fluids. This investigation will be concerned with the damping characteristics obtained for the incompressible squeeze-film damper. Such a damper appears in Fig. 1, which shows a rolling-element bearing mounted in an oil squeeze-film damper. The annulus between the outside diameter of the ball bearing housing and the inside diameter of the damper housing is filled with oil. The orbital motion or precession of the ball bearing housing in the damping fluid generates a hydrodynamic pressure. This particular type of damper is now being used in several production aircraft turbojet engines and in other high-speed turbomachinery.

The purposes of this investigation are to (1) examine the influence of flexible damped supports on rotor amplitudes and forces transmitted over a speed range encompassing several critical speeds, (2) show how single-mass rotor theory can be used to de-
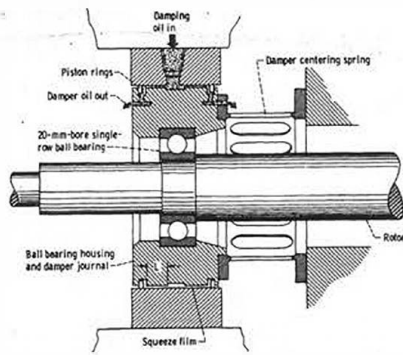
Fig. 1 Squeeze-film damper with central feed groove

Fig. 2 Single-mass flexible rotor

$$\omega_{cr} = \sqrt{K_2/M_2} \qquad (3)$$

where K_2 is the equivalent rotor and bearing stiffness given by

$$K_2 = \frac{K_b K_s (K_s + K_b) + K_s(\omega B_b)^2}{(K_b + K_s)^2 + (\omega B_b)^2} \qquad (4)$$

When the bearing damping is small (as in rolling element bearings), the equivalent stiffness and damping reduce to

$$K_2 = \frac{K_b K_s}{K_b + K_s} \qquad (5)$$

$$B_2 = \frac{K_s^2 B_b}{(K_b + K_s)^2} + B_s \qquad (6)$$

Equations (1)–(6) are taken from reference [3].

Damping Required to Attenuate Rotor Amplitude at First Critical Speed. The rotor of Fig. 2 is modeled as a mass of 2.42 kg on a massless elastic shaft. The spring rate for this rotor is calculated from

$$K_s = \frac{3\pi E D_s^4}{4 l^2}$$

where

$$
\begin{aligned}
E &= 210 \text{ GN/m}^2 \\
D_s &= 25.4 \text{ mm} \\
l &= 480 \text{ mm}
\end{aligned}
\qquad (7)
$$

Thus

$$K_s = 1.82 \text{ MN/m}$$

The rotor is mounted in ball bearings which are assumed to have a much higher stiffness than that of the shaft. Thus, the critical speed on rigid supports can be calculated from

$$\omega_{cr} = \sqrt{K_2/M_2} = \sqrt{K_s/M_2}$$

$$\omega_{cr} = 867 \text{ rad/s (8280 rpm)}$$

The amplification factor A for this rotor on rigid supports is assumed to be 10. Solving equation (1) for B_2

$$B_2 = 210 \text{ N-s/m}$$

Fig. 3 has been plotted for a mass ratio M of 1, which is the ratio of bearing housing mass M_1 to rotor mass M_2. Fig. 4 (from reference [3]) shows that, if one has a choice of stiffness ratios, the mass
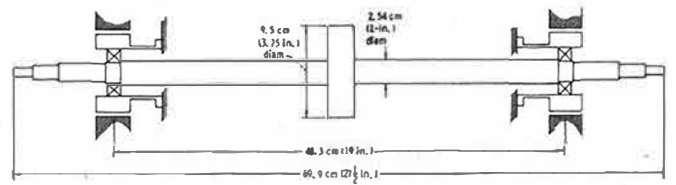
sign a support system for a multimass rotor operating below the second bending critical speed, and (3) demonstrate design procedure for an oil squeeze-film damper.

Influence of Damper Support on Single-Mass Flexible Rotor

Fig. 2 shows a single-mass rotor mounted in elastic damped supports. Fig. 3 represents the rotor amplitude for various values of support damping for this system. In Fig. 3 rotor speed has been normalized with respect to the rigid support critical speed. If the bearing supports have little damping ($B \le 2$), the rotor then will have a very high response at the critical speed. For this case the critical speed is about $\%_{10}$ of the rigid support critical speed. Also, at high operational speeds, this rotor-bearing system is highly susceptible to self-excited whirl motion.

As the damping is increased in the flexible support, the rotor amplitude at the critical speed is diminished until, at the optimum value of dimensionless support damping ($B = 14$), the motion at the critical speed is completely damped out. However, if the support damping value is increased beyond the optimum value, the amplitude of motion will increase at the rigid support critical speed. For example, a damping value of 50 is excessive and causes the support to "lock up." Fig. 3 has been plotted for an amplification factor A of 10. Amplification factor is defined as the ratio of the rotor amplitude at the rotor critical speed to the rotor unbalance eccentricity; $A = a_{cr}/e_\mu$. A value of 10 represents moderately light damping. Reference [3] points out that optimum support damping is virtually independent of A for $A \ge 10$; thus, the information of reference [3] is applicable to a wide range of rotors.

Amplification factor can also be expressed as

$$A = K_2/(\omega_{cr} B_2) \qquad (1)$$

where B_2 is the equivalent rotor and bearing damping given by

$$B_2 = \frac{K_s^2 B_b}{(K_b + K_s)^2 + (\omega B_b)^2} + B_s \qquad (2)$$

The critical speed ω_{cr} is calculated from

Nomenclature

A = amplification factor at rotor critical speed, dimensionless
a_1 = support amplitude, m
a_2 = rotor amplitude, m
B = damping ratio B_1/B_2
B_b = bearing damping, N-s/m
B_d = squeeze film damping, N-s/m
$\bar{B}_d$ = dimensionless squeeze film damping, $B_d c^3/\mu R L^3$
B_s = shaft damping, N-s/m
B_1 = total support damping, N-s/m
B_2 = effective rotor-bearing system damping, N-s/m
c = radial clearance in damper bearing, m

D_s = shaft diameter, m
E = elastic modulus, N/m^2
e = bearing eccentricity, m
e_μ = rotor mass eccentricity, m
F_b = bearing load, N
F_1 = force transmitted to foundation, N
K = stiffness ratio K_1/K_2
K_b = bearing stiffness, N/m
K_c = centering spring stiffness, N/m
K_d = squeeze film stiffness, N/m
$\bar{K}_d$ = dimensionless squeeze film stiffness, $K_d c^3/\mu \omega R L^3$
K_s = shaft stiffness, N/m
K_1 = total support stiffness, N/m

K_2 = effective rotor-bearing system stiffness, N/m
L = damper half-length, m
l = bearing span, m
M = mass ratio M_1/M_2
M_1 = total support or bearing housing mass, kg
M_2 = rotor mass, kg
R = damper bearing radius, m
U = unbalance, kg-m
ϵ = eccentricity ratio e/c
μ = damper fluid viscosity, N-s/m^2
ω = angular velocity, rad/s
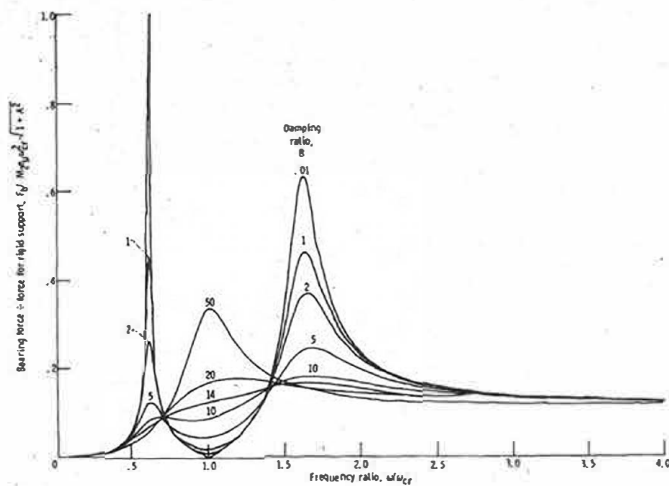ω_{cr} = rigid support critical speed, rad/s

Fig. 7 Dimensionless force transmitted to bearings as function of speed ratio for various values of support damping (from reference [3]). Stiffness ratio, 1; mass ratio, 1; amplification factor, 10.

bending critical speed. Oil squeeze-film dampers are used at the bearing supports to attenuate rotor motion. Two single-row, deep groove ball bearings, series 204, support the rotor. The bearing support housings each have a mass of 1.21 kg. A cantilevered centering spring supports the ball bearing housing; the spring rate can be chosen to complement that of the squeeze film. Oil is supplied to the damper from a central circumferential groove. Two piston rings located in circumferential grooves along with metering orifices serve to control the flow of damping oil from the bearing ends. These features are shown in Fig. 1.

Critical speeds were calculated for this rotor, on rigidly supported bearings, by the critical speed computer program of reference [4]. A bearing stiffness of 65.5 MN/m was assumed. The first three critical speeds and associated mode shapes are shown in Fig. 9. The first bending critical speed was calculated as 867 rad/s (8280 rpm).

The shaft and bearing span of this five-mass rotor are identical to those of the single-mass rotor discussed previously. Thus, the five-mass rotor will have the same stiffness as the single-mass rotor. For calculation purposes, the five masses may be replaced with a single mass at the rotor center which will result in the same critical speed. This equivalent single mass may be calculated from

$$M_2 = K_2/\omega_{cr}^2 = 2.42 \text{ kg}$$

This contrasts markedly with the actual rotor mass of 5.68 kg.

The equivalent mass is identical to that of the single-mass rotor discussed previously. The ratio of support mass to rotor mass is

$$M = \frac{M_1}{M_2} = \frac{2(1.21)}{2.42} = 1$$

The supports will be designed for a tuned system, that is, $K = M$. This being the case, the rotor properties, support stiffness ratio, and mass ratio are the same as those for the single-mass rotor discussed previously, and all of the previously calculated values may be used without change. Thus, the required damping ratio B is 10–14 and the support damping B_1 is 2100 to 2940 N-s/m, or one-half of this for each of the two supports.
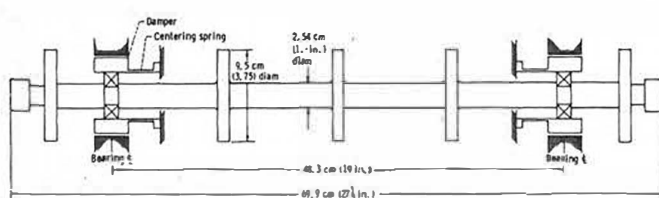

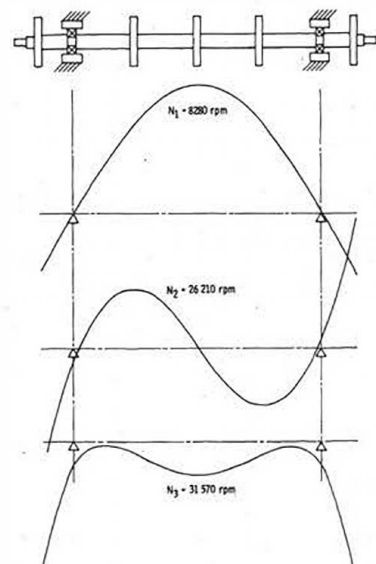
Fig. 8 Five-mass flexible rotor

Fig. 9 Undamped critical speeds and mode shapes of five-mass rotor on rigid supports. Bearing stiffness, 65.5 MN/m (375,000 lb/in.).

Design of Squeeze-Film Damper

A squeeze-film damper is depicted schematically in Fig. 1. It consists of a cylindrical journal, which is prevented from rotating, in a cylindrical bearing. The journal center is assumed to make a circular orbit about the bearing center.

Reference [5] solves the Reynolds lubrication equation to obtain the forces acting on the bearing. The analysis assumes full cavitation, that is, negative pressures in the oil film are ignored. (A fluid cannot generally sustain a tensile stress.) The results may be expressed in terms of dimensionless stiffness and damping coefficients as

$$\bar{K}_d = \frac{K_d c^3}{\mu \omega R L^3} = \frac{2\epsilon}{(1 - \epsilon^2)^2} \tag{8}$$

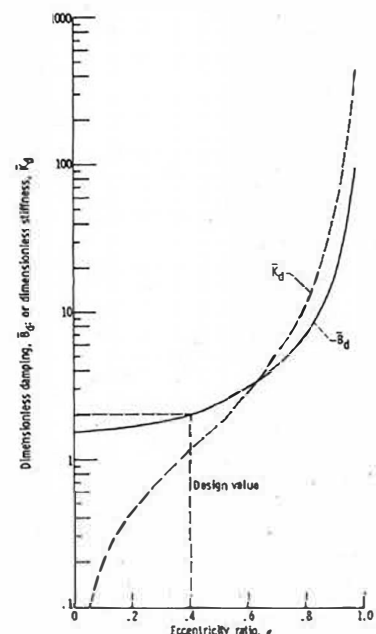$$\bar{B}_d = \frac{B_d c^3}{\mu R L^3} = \frac{\pi}{2(1 - \epsilon^2)^{1.5}} \tag{9}$$

Fig. 10 Dimensionless damping (equation (9)) and stiffness (equation (8)) as functions of eccentricity ratio

Both $\bar{K}_d$ and $\bar{B}_d$ are functions only of the eccentricity ratio and are plotted as such in Fig. 10.

The magnitude of damping required has been determined, and it is now necessary to design the damper bearing to produce this amount of damping. Equation (9) shows that damping B_d is a function of damper clearance, length, radius, eccentricity ratio, and oil viscosity. Generally, the diameter of the damper housing is dictated by the ball bearing outside diameter. The same oil is usually used for both the damper and the ball bearing; thus, the viscosity of the damper oil is fixed. It remains, however, to select values of the radial clearance c and half-length L.

Since damper stiffness increases rapidly with increasing eccentricity ratios (equation (8)), it is not desirable to operate at very large eccentricity ratios because the film stiffness will make the overall support stiffness too large. A maximum eccentricity ratio of $\epsilon = 0.4$ at the first critical speed was chosen for this damper design. For $\epsilon = 0.4$, a value of dimensionless damping $\bar{B}_d$ of 2.04 is obtained from Fig. 10.

The next parameter to determine is the damper clearance c. Reference [3] shows that for an optimally damped system, the maximum support amplitude is about equal to the mass eccentricity ϵ_μ of the rotor, that is, the displacement of the rotor center of gravity from the geometrical center. The five-mass rotor of Fig. 8 is expected to have a maximum unbalance of 29 g-cm distributed fairly uniformly over the five masses. This corresponds to a mass eccentricity of 0.05 mm. If the maximum damper eccentricity ratio is to be 0.4, the damper clearance must be 0.05/0.4 = 0.13 mm.

It remains only to determine the damper length. The design of Fig. 1, with a circumferential oil supply groove, divides the damper into two equal halves. Each half functions separately and thus provides one-fourth of the total required support damping. The total support damping needed was previously determined to be 2100–2940 N-s/m. The rotor is expected to operate with considerably less than the maximum unbalance most of the time; the maximum 29 g-cm represents a degraded value after considerable service time. Less unbalance means lower amplitudes, and thus lower stiffness of the squeeze film (equation (8)). To maintain squeeze-film stiffness as much as possible at lower eccentricities, the damper will be sized for damping near the top of the range for an eccentricity ratio of 0.4. Thus, the design value will be 2800 N-s/m. Damper length can now be determined from

$$L = c\sqrt[3]{\frac{B_1}{4\bar{B}_d\mu R}}$$

in which

$$c = 0.13 \text{ mm}$$
$$\mu_{60°C} = 0.0119 \text{ N-s/m}^2$$
$$R = 39.6 \text{ mm}$$
$$B_1 = 2800 \text{ N-s/m}$$
$$\bar{B}_d = 2.04$$

Length L is then 11.4 mm.

The net radial stiffness of the cavitated squeeze film can now be calculated. For $\epsilon = 0.4$, a value of dimensionless stiffness $\bar{K}_d = 1.13$ is obtained from Fig. 10. A value of film stiffness can now be calculated from

$$K_d = \frac{\bar{K}_d \mu \omega R L^3}{c^3}$$

where ω is the damper angular precession speed. At the first critical speed of 8280 rpm (867 rad/s)

$$K_d = 0.338 \text{ MN/m}$$

The total damper spring rate K_1 is four times this value, or 1.35 MN/m. As was stated before, the overall support stiffness is the sum of the centering spring stiffness and the damper stiffness since the two springs act in parallel. The centering spring stiffness may now be chosen to provide optimum stiffness at the first critical speed. For $K = 1$, total support stiffness $K_1 = 1.82$ MN/m.

Thus the stiffness K_c of each centering spring must be (1.82 − 1.35)/2 = 0.235 MN/m. This is a very soft spring and will need to be preloaded to center the rotor at low speeds.

Performance of Flexibly Supported Multimass Rotor

To determine how effective the damped flexible support is in attenuating the rotor amplitude and bearing forces, an unbalance response computer program was used to produce Figs. 11–13. The program is that of reference [6] modified for (1) axisymmetric rotor supports and (2) nonlinear stiffness and damping in the squeeze film.

Fig. 11 shows rotor amplitude at midspan and Fig. 12 shows forces that would be transmitted to the ball bearing for both a rigid support and for a flexible damped support. The long-dash curves of these figures represent the rotor operating with good balance; the total unbalance U is 7 g-cm. The corresponding mass eccentricity $\bar{e}_\mu$ is 0.012 mm. The first critical speed has shifted upward to about 9000 rpm.

Fig. 11 shows that the rotor amplitude in this case is about three times the mass eccentricity at the first critical speed. The amplitude then drops with increasing speed and remains low out to 30000 rpm. Amplitude then increases with speed, reaching a maximum of nearly five times the mass eccentricity at 39000 rpm. Fig. 12 shows that the bearing experiences virtually no force buildup due to the critical speed. At 9000 rpm the bearing force is only 14 N. This is only one-half the force that would be experienced by a bearing on a rigidly supported rigid rotor (for which $F_b = (\frac{1}{2})M_2 e_\mu \omega^2$). Bearing force generally rises with speed to 39000 rpm and then drops off.

Now consider the case in which the initial balance at assembly has degraded for any one or a combination of reasons mentioned in the Introduction. Instead of a total unbalance of 7 g-cm let us assume the unbalance now is 29 g-cm resulting in a mass eccentricity of 0.05 mm. The solid curves of Figs. 11 and 12 show results for the rotor on squeeze-film supports, while the dashed curves show results for the rotor on rigid supports. At the first critical speed, the center amplitude of the flexibly supported rotor is about three times the mass eccentricity, as with the lesser unbalance. Over the entire speed range the amplitude change is nearly the same as for the low unbalance. However, the peak amplitude speed has shifted from 39000 to 43000 rpm due to greater damping in the squeeze film at the higher eccentricity. The rigidly supported rotor, in contrast, has an amplitude of 50 times the mass eccentricity at the first critical speed. Thus the squeeze film support reduces the maximum motor amplitude by a factor of 16.

Fig. 12 shows that bearing force F_b at the first critical speed is 51 N with the flexible support, whereas with a rigid support it is 1900 N, or 36 times greater. Similar load reductions occur at the higher critical speeds. At 28000 rpm, the load for the rigidly supported bearing is 26000 N. If the 204 size ball bearing is to operate for 2000 hr at this speed with a reliability of 99 percent, the load cannot exceed 1530 N [7–9]. This load is greatly exceeded when the bearing is rigidly supported. With the squeeze-film support, however, the load is only 480 N, well within the bearing capacity.

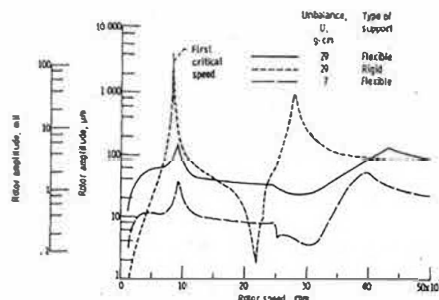Fig. 13 shows the resultant damper amplitudes for the flexibly

Fig. 11 Amplitude at rotor midspan for rigidly and flexibly supported five-mass rotor
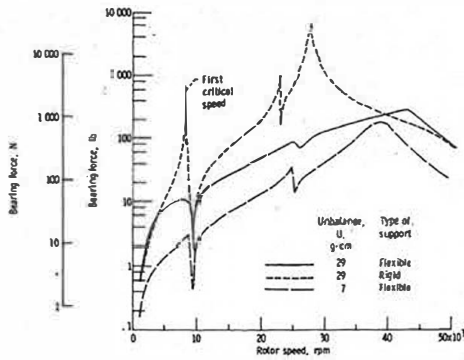
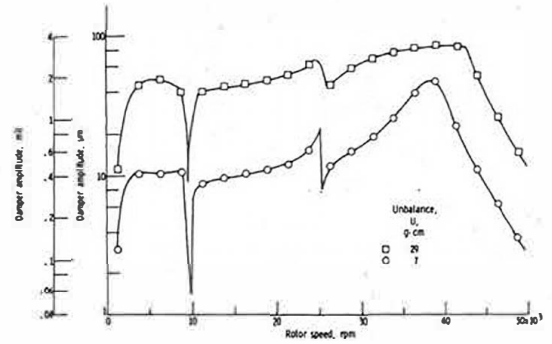Fig. 12 Bearing forces for rigidly and flexibly supported five-mass rotor

Fig. 13 Damper amplitude for flexibly supported rotor

supported rotor. As predicted for a single-mass rotor, damper amplitudes are approximately equal to the mass eccentricity up to twice the first critical speed. At higher speeds, damper eccentricities increase. For the large unbalance, the increase is about 70 percent, and for the small unbalance, by a factor of nearly 5.

Though the damper was sized only for the first critical speed, the results show that amplitudes and forces at the higher critical speeds are also reduced substantially. Thus, single-mass rotor data are useful, not only for multimass rotors operating below the second critical speed, but also at higher speeds. However, for rotors operating through several critical speeds, the authors recommend that a rotor response analysis be used, after design of the damper, to determine damper and rotor performance at the higher critical speeds.

Rotor response was also calculated with the 29 g-cm unbalance concentrated at the center mass, rather than distributed over the five masses. This resulted in a much larger unbalance loading than with distributed unbalance. Consequently, the squeeze-film damper was overloaded; rotor amplitudes and bearing loads were approximately double those for rigidly supported bearings. The point to be noted is that an improperly designed squeeze-film damper (inadequate clearance, etc.) can be worse than a rigid bearing support. Reference [10] also illustrates this phenomenon.

Summary of Design Procedure

1   For a symmetric multimass rotor, determine an equivalent single mass M_2 from

$$M_2 = \frac{K_2}{\omega_{\mathrm{cr}}^2}$$

where K_2 is the rotor-bearing stiffness at midspan (given by equation (4)) and ω_{cr} is the first bending critical speed for the multimass rotor on rigidly supported bearings.

2   Calculate the mass ratio $M = M_1/M_2$ for the total bearing mass M_1 to be used and the equivalent rotor mass M_2.

3   From Fig. 5 determine the optimum damping ratio B. Determine the absolute support damping B_1 required from $B_1 = B \times B_2$. For lightly damped bearings, the value of effective bearing system damping B_2 may be estimated from equation (1) with $A = 10$:

$$B_2 = K_2/10\omega_{\mathrm{cr}}$$

4   Determine the absolute support stiffness K_1 from

$$K_1 = K \times K_2$$

For a tuned system, which produces near-minimum rotor amplitude, $K = M$. Fig. 5 and the design examples of this report assume that a tuned system is to be used.

5   Assume a damper eccentricity ratio ϵ that will not make the overall stiffness of the support too large, and from Fig. 10 determine values of dimensionless stiffness $\bar{K}_d$ and damping $\bar{B}_d$. Generally, the maximum value of ϵ should be 0.4 or less.

6   From the actual rotor mass and the maximum anticipated unbalance, calculate the mass eccentricity from

$$e_\mu = \frac{U}{M_2}$$

For an optimally damped system, the maximum support amplitude is approximately equal to the mass eccentricity, that is, $a_1/e_\mu = 1$.

7   The damper clearance can now be calculated from

$$c = \frac{e_\mu}{\epsilon}$$

8   For a circumferentially grooved damper, the damper half-length is determined from the following:

$$L = c \sqrt[3]{\frac{B_1}{4\bar{B}_d\mu R}}$$

The values of total support damping B_1, the squeeze damping $\bar{B}_d$, and the clearance c have been determined in steps 3, 5, and 7. The damper radius R will usually be dictated by the size of the rolling or sliding bearing to be used. The viscosity μ is that of the bearing lubricating oil.

9   Calculate the stiffness of each centering spring from

$$K_c = (1/2)(K_1 - 4\bar{K}_d\mu\omega_c RL^3/c^3)$$

Summary of Results

Theoretical data for a single-mass rotor were used to determine flexible support properties (stiffness and damping) to attenuate rotor amplitudes and bearing loads for a multimass rotor operating through the first bending critical speed. An equivalent single mass for the multimass rotor was calculated from the rotor first critical speed (determined from a critical speed computer program) and the rotor shaft stiffness. A squeeze-film damper support was then designed to provide the required damping at the assumed unbalance conditions. Analytical rotor response results showed that

1   The squeeze-film damper successfully attenuated rotor amplitudes and bearing loads at the first critical speed. Rotor midspan amplitude was reduced by a factor of 16, and bearing load was reduced by a factor of 36 compared with an identical rotor having rigidly supported bearings.

2   Amplitudes and forces at higher critical speeds were also reduced substantially.

3   With unbalance less than the design value, amplitudes and forces were also well controlled. However, with unbalance much greater than the design value, amplitudes and forces were larger than with rigidly supported bearings.

4   Bearing loads were well under permissible values for the flexibly supported rotor. With rigid supports, bearing forces were very high near rotor critical speeds, resulting in substantially overloaded bearings.

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