Lund's Contribution to Rotor Stability: The Indispensable and Fundamental Basis of Modern Compressor Design

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In the early design of compressors and turbines, it was of importance to locate the rotor critical speeds so as to insure that a turbo rotor would not be operating at a critical speed. As compressor and turbine design became more sophisticated, a more detailed analysis of the rotor damped critical speeds and rotor log decrements was necessary. With compressors and turbines operating at higher speeds, under high power levels, and at multiples of the first critical speed, the problem of stability or self-excited whirling is of paramount importance. The onset of self-excited motion may lead to large amplitudes of motion with possible destructions of the rotor or the inability of the compressor to operate at peak power levels. Encountering this phenomenon with a compressor on an off-shore oil platform can mean millions of dollars of lost production. Self-excited whirling can be caused by fluid-film bearings, seals, Alford-type cross-coupling forces, or internal shaft friction, to name a few of the excitation mechanisms. The problem of computing the approximate values of the rotor log decrement under full speed and loading conditions requires the solution of a complex eigenvalue problem. The computation of the rotor complex roots is an order of magnitude more difficult than the problem of undamped critical speed calculations. J. W. Lund presented the first practical numerical procedure for computing turbo-rotor log decrements. The mathematical transfer matrix method pioneered by Lund has allowed industry to develop and stabilize a vast array of rotating machinery leading to the savings to industry of millions of dollars. Without the procedures of Lund, for example, it would not have been possible to resolve stability problems encountered with both the hydrogen and oxygen space shuttle pumps. This paper briefly presents some of the attributes of the Lund stability procedure and its unique characteristics. [DOI: 10.1115/1.1605978]

Introduction

This paper is written in tribute to Jørgen W. Lund, who may be looked upon as the father of modern rotor dynamics. Over the years, I have kept a collection of Lund's various papers. Lund has presented significant papers on gas bearings, multi-lobed fluid-film and tilting-pad bearings, unbalance response, rotor-bearing stability and balancing, and transient rotor response. In this paper, I wish to discuss the significant contributions that Lund has made towards the predictions of stability of turbo-machinery. Practical computer programs based on the theories and numerical methods presented by Lund in rotor bearing dynamics have literally revolutionized the design of modern turbo-rotors.

It may be difficult for current designers to appreciate the contributions made by Lund in the field of rotor-bearing stability. Now, there exist numerous commercial programs available for stability analysis based upon both the transfer matrix and the finite element approaches. The API Code now requires that lateral, torsional, and damped eigenvalues be computed for new compressor designs.

There are two papers in particular that I wish to comment on and discuss in detail. The first paper is "Stability and Damped Critical Speeds of a Flexible Rotor in Fluid-Film Bearings," 1974 [1] and the other is his bearing paper with K. K. Thompson on "A Calculation Method and Data for the Dynamic Coefficients of Oil-Lubricated Journal Bearings," 1978 [2]. The first paper sets forth the practical solution for the stability or damped eigenvalue analysis of turbo-rotors for the first time using a digital computer. Using the Lund procedure, it was now possible to compute rotor stability for eight- and eleven-stage compressors with multiple fluid-film bearings, seals, and aerodynamic Alford-type cross-coupling effects.

In order to accurately compute the stability characteristics of a compressor or turbine, one must know the stiffness and damping coefficients for the fluid-film bearings. Next to Osborne Reynolds' original paper, "On The Theory of Lubrication and Its Application to Mr. Tower's Experiments," published in 1886 in Phil. Trans. Society of London [3], I consider Lund's paper with K. K. Thompson to be one of the most comprehensive papers written on fluid-film bearings. The reason for this is twofold: First, a numerical procedure is presented for the analysis of load and dynamic coefficients of finite width fluid-film bearings; and second, extensive bearing data is presented on the axial and various multi-lobed bearings for various aspect ratios based upon interpolation of the Lund data. I should not fail to mention the significant publications by Lund on tilting-pad bearings, as well [4]. His contribution in tilting-pad bearings will be addressed in more detail by Dr. John Nicholas [5]. Any serious student of rotor dynamics should not only study these papers in detail, but should review the derivations of the governing equations for stability and bearing analysis.

Introduction to Stability and Computation Problems. In order to better illustrate the contribution by Lund to stability analysis of turbo-machinery, we will examine several simple systems and their governing equations of motion. Some of the difficulties and complications involved with the solution of damped

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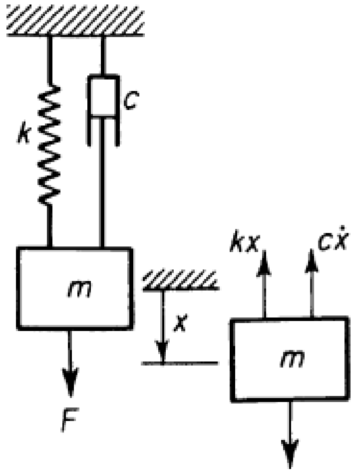

Fig. 1 Single-degree-of-freedom system

eigenvalues will be pointed out. It will later be presented how the transfer matrix method of Lund avoids many of these difficulties.

Single-Mass System. The simplest damped eigenvalue problem starts with the single-mass, single degree-of-freedom system shown in Fig. 1. This is the classical mechanical system, as shown in every elementary vibrations text

$$M\frac{d^2X}{dt^2}+C\frac{dX}{dt}+KX=0 \tag{1}$$

Assume a solution of the form

$$X(t)=Ae^{\lambda t}$$

This results in

$$(M\lambda^2+C\lambda+K)A=0 \tag{2}$$

Dividing by M, we obtain

$$\lambda^2+2\xi\omega_c\lambda+\omega_c^2=0 \tag{3}$$

here

$$\xi=\frac{C}{C_c},\quad C_c=2M\omega_c$$

$$\omega_c=\sqrt{\frac{K}{M}}$$

Equation (3) is referred to as the characteristic equation for the system. From the examination of the coefficients of the characteristic polynomial, we may gather a great deal of insight into the nature of the stability problem. We will also understand some methods which are not appropriate for stability analysis.

Equation (3) is second order and may be factored into its roots as follows:

$$(\lambda-\lambda_1)(\lambda-\lambda_2)=0 \tag{4}$$

The simple solution to the 2nd order characteristic polynomial of Eq. (3) is the classical solution where the complex eigenvalue is:

$$\lambda_{1,2}=p\pm i\omega_d,\quad p=-\xi\omega_c$$

Where

$$p=\text{real part}=\frac{-C}{2M}=-\xi\omega_c$$

$$\omega_d=\text{damped natural frequency}=\omega_c\sqrt{1-\xi^2}$$

The amplitude

$$X(t)=e^{pt}[A\cos\omega_d t+B\sin\omega_d t]$$

For a system of n degrees-of-freedom, the size of the characteristic polynomial would be $2n$, as follows

$$(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_{2n-1})(\lambda-\lambda_{2n})=0 \tag{5}$$

For the case of Eq. (4), we obtain

$$\lambda^2-(\lambda_1+\lambda_2)\lambda+\lambda_1\lambda_2=0 \tag{6}$$

For Eq. (5), we would obtain

$$\lambda^{2n}-(\lambda_1+\lambda_2+\ldots\lambda_{2n})\lambda^{2n-1}+\ldots+\pi_{r=1}^{2n}\lambda_r=0 \tag{7}$$

The coefficients $\Sigma_{r=1}^{2n}\lambda_r$ and $\pi_{r=1}^{2n}\lambda_r$ are called the invariants of the characteristic polynomial. We now observe an interesting feature of the characteristic polynomial. The first coefficient, or invariant, is the sum of the eigenvalues, and the last invariant is equal to the product of all the eigenvalues. By examining the invariant coefficients of the characteristic polynomial governing the stability problem, we are struck by two apparent facts. First, for a system of n degrees-of-freedom, the characteristic polynomial will be of order $2n$, and that the last coefficient, or invariant, will be a product of $2n$ eigenvalues. From a practical standpoint, this means that without proper scaling, any rotor system beyond 10th order (which is a very small rotor model) will begin to encounter numerical roundoff errors. With proper scaling, one can extend the system to order 20.

Thus, we see that for a simple linear system the motion is stable or damped out when the real root p is negative, is at the threshold of stability when $p=0$, and is unstable when $p>0$. Near the threshold of stability, the value of the real component p is small in comparison to the damped natural frequency ω_d. Thus, accurate computations of the stability threshold can be a difficult numerical problem for a large multi-mass system.

In stability, we do not refer to the magnitude of the real root p, but rather to the log decrement defined by

$$\ln\left(\frac{X(t)}{X(t+\tau)}\right)=\frac{2\pi\xi}{\sqrt{1-\xi^2}}=\delta \tag{8}$$

For low values of the damping ratio ξ, the amplification factor at the critical speed is given by

$$A_c=\frac{1}{2\xi}\approx\frac{\pi}{\delta} \tag{9}$$

Thus, a stable machine with a log decrement of 0.1 would give a critical speed amplification factor of over 30. Accurate calculation of the rotor log decrement is important not only from a stability standpoint, but also to insure that the rotor critical speed amplification factors are reasonable.

Single-Mass Flexible Rotor. Figure 2 is the classical single-mass Jeffcott rotor on simple supports. The center line, C, is whirling about the origin, o. For steady-state motion two degrees-of-freedom are required to express the motion.

It would appear to be natural to express the equation of motion in polar coordinates, with the radial motion δ representing the elastic deflection of the rotor and ϕ the precession rate. This seems particularly desirable when including plain journal bearings. The use of polar coordinates is not a desirable approach, as it leads to nonlinear dynamical equations of motion.

One of the earliest papers on rotor dynamics was W. A. Rankine ''On the Centrifugal Force of Rotating Shafts,'' in 1869. Rankine used a rotating coordinate system similar to Fig. 2 and arrived at the following equation of motion for the radial direction, as shown in Ref. [3].

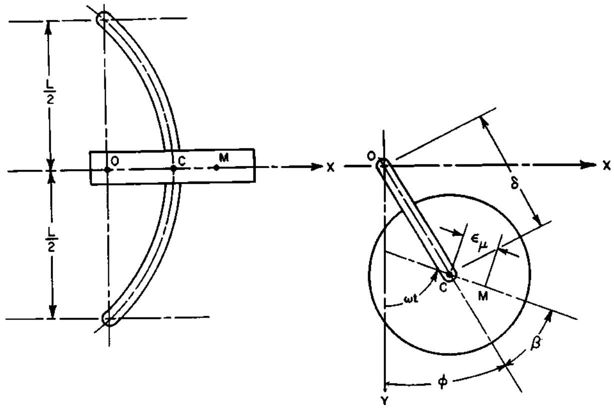

Fig. 2 Single-mass rotor

$$\ddot{\delta}+\frac{C}{M}\dot{\delta}+(\omega_c^2-\dot{\phi}^2)\delta=e\omega^2\cos\beta \tag{10}$$

Thus, he arrived at the conclusion that when the rotor speed ϕ was below the critical speed ω_c, the rotor was stable, in "indifferent" equilibrium at the critical speed, and was unstable above the critical speed. The problem lies in the neglect of the Coriolis terms of the second equation of motion. Papers are still being presented today in which the author attempts to solve the dynamics problem in rotating coordinates. This problem was correctly solved by Jeffcott in 1919 when he used a fixed Cartesian coordinate system. Lund makes use of the fixed Cartesian coordinate system in his general flexible rotor stability analysis. With the two degree-of-freedom Jeffcott rotor on simple supports, the situation of self-excited whirl motion may be caused by the internal shaft damping C_r or the aerodynamic cross-coupling ϕ acting at the turbine wheel. Assuming there is external damping C acting at the disk, the two coupled equations of motion are given by

$$M\ddot{X}+(C+C_i)\dot{X}+KX+(Q+\omega C_i)Y=0 \tag{11}$$

$$M\ddot{Y}+(C+C_i)\dot{Y}+KY-(Q+\omega C_i)X=0 \tag{12}$$

The two equations form a 4th order characteristic polynomial. At the time I was analyzing the Jeffcott rotor for my thesis research, I could determine the stability boundary by the use of the Routh Hurwitz criterion and could even integrate the equations of motion with an analog computer, but I could not determine the log decrement of even this simplest of systems. It was not possible to compute the log decrement until Lund presented his stability method.

Extended Jeffcott Rotor. The single-mass Jeffcott rotor as shown in Fig. 2 does not simulate a realistic rotor since it has pinned bearings. The second major contribution Lund made to the ability to compute compressor or turbine log decrements was the publication of various papers on gas bearings, as well as oil-lubricated bearings. Reference [2] by Lund on fixed-lobed bearings is of particular significance, as already mentioned, and has bearing coefficients listed for L/D ratios of 0.5 and 1.0.

Figure 3 represents the extended Jeffcott rotor on flexible supports. This model may be effectively applied to first mode stability log decrement studies by assuming the center mass M to be the rotor multi-station modal mass and using the modal compressor stiffness as K. The Lund bearing coefficients may then be used to rapidly determine the stability characteristics for various bearing types.

The equations of motion are expressed in fixed X, Y Cartesian coordinates. For a multi-mass rotor it is not feasible or desirable to employ a rotating coordinate system. More will be said about this approach later. The three X and Y displacements are

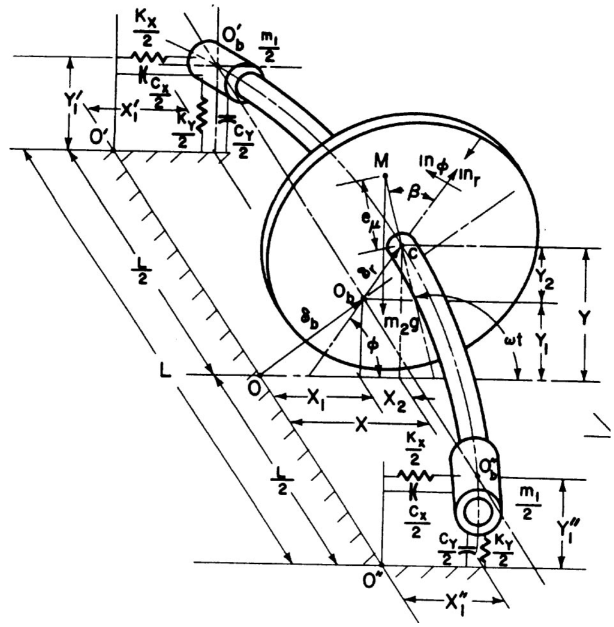

Fig. 3 Extended Jeffcott rotor on flexible supports

$$\{X\}^T=\{X_1X_2X_3\}^T$$

and

$$\{Y\}^T=\{Y_1Y_2Y_3\}^T$$

The six equations of motion are of the form

$$\begin{bmatrix} M & \\ & M \end{bmatrix}\begin{Bmatrix}\ddot{X}\\ \ddot{Y}\end{Bmatrix}+\begin{bmatrix} C_{xx} & C_{xy}\\ C_{yx} & C_{yy}\end{bmatrix}\begin{Bmatrix}\dot{X}\\ \dot{Y}\end{Bmatrix}+\begin{bmatrix} K_s & \\ & K_s\end{bmatrix}\begin{Bmatrix}X\\ Y\end{Bmatrix}+\begin{bmatrix} K_{xx} & K_{xy}\\ K_{yx} & K_{yy}\end{bmatrix}\begin{Bmatrix}X\\ Y\end{Bmatrix}=0 \tag{13}$$

Each item in Eq. (13) represents a 3×3 matrix. The matrix K_s is of particular interest, as this represents the effective finite element stiffness matrix of the shaft and is given by

$$[K_s]=\begin{vmatrix} \frac{K}{4} & -\frac{K}{2} & \frac{K}{4}\\ -\frac{K}{2} & K & -\frac{K}{2}\\ \frac{K}{4} & -\frac{K}{2} & \frac{K}{4}\end{vmatrix}$$

where $K=\text{shaft stiffness}=48EI/L^3$ for a simple shaft. As a finite element three-point stiffness matrix, the matrix must be positive, definite, and singular to represent a free-free shaft. The 12th order characteristic polynomial for this system may be generated by the Leverrier's algorithm, as given in Ref. [6]. This method of stability analysis is not practical for a large multi-mass compressor, as the shaft nth order stiffness matrix would have to be obtained from structural analysis, and numerical difficulties in the root solving procedures would limit its utility.

It is of interest to note that a number of papers have been generated on the extended Jeffcott rotor with one end fixed, such as with a ball bearing. In this case the constrained shaft stiffness matrix becomes

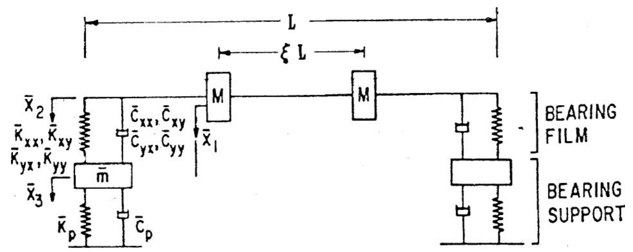

Fig. 4 Lund 2-mass rotor on damped supports

$$K_s=\begin{bmatrix} \frac{K}{4} & -\frac{K}{2} \\ -\frac{K}{2} & K \end{bmatrix}$$

In the majority of papers on this subject, the $K/4$ term is missing from the equations, leading to misinterpretation of the results such as assumed fluid film bearing inertia terms. It is apparent that even the simple Jeffcott rotor, when extended to flexible supports, can lead to difficulties due to errors in the system equations. The attempt to add several mass stations by this approach leads to problems unless the corresponding shaft stiffness matrix is properly derived by finite element techniques or the inversion of a flexibility matrix.

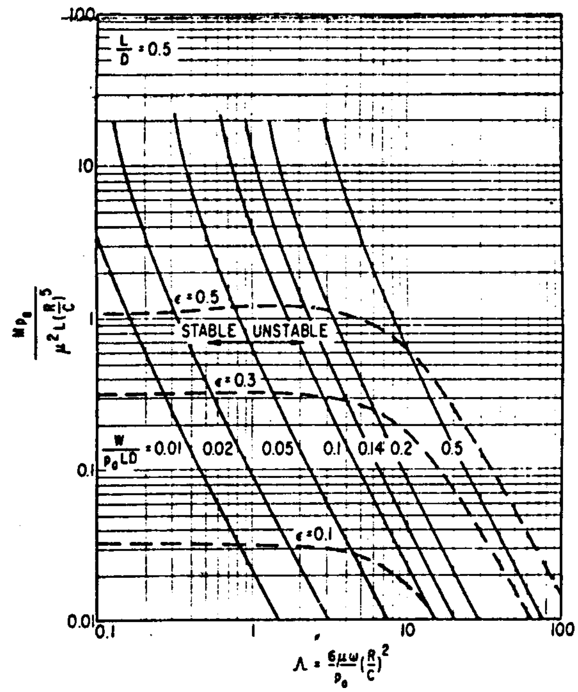

Fig. 5 Critical rigid rotor mass for $L/D=0.5$ gas bearing vs. dim. speed

Lund on Stability of Gas Bearings

The earliest works of Lund on rotor-bearing stability were concerned with static and dynamic characteristics of gas bearings. In the 1960's there was a considerable research effort performed on gas-lubricated bearings, both in the United States and abroad. The research was broadly supported by the Office of Naval Research (ONR), the NEC, NASA Lewis Research Center, and the Wright Patterson Air Force Laboratories. Extensive government-funded gas bearing research programs were conducted at Ampex under the direction of Dr. William Gross, at Franklin Institute under Professor Dudley Fuller, and at Mechanical Technology, Inc. under the direction of Dr. Beno Sternlicht. Lund, who was at MTI during this time, made major contributions to the field of gas bearing dynamics and stability. Dr. Sternlicht had assembled some of the best engineers in the country at MTI in the 1960's, and Lund was considered to be one of his most outstanding stars.

The solution of the Reynolds Equation for a gas bearing is complicated by the compressibility of the lubricant. This makes the Reynolds Equation for gas-lubricated bearings nonlinear in the pressure term P. One of Lund's first analyses of flexible rotor stability was in 1965 in his paper, "The Stability of an Elastic Rotor in Journal Bearings With Flexible, Damped Supports." Figure 4 represents the rotor model used by Lund [6].

If journal mass is included in this model, then the order of the system would be 12. This would be beyond analysis capabilities at the time, even for Jørgen. Simplifications made by Lund included neglecting journal mass, and later support mass, in his analysis. The shaft stiffness matrix for general motion in Fig. 4 should be 4×4, Lund assumes that the motion of the two major masses is either in phase (1st mode) or out-of-phase (2nd mode). Representing the rotor by flexibility coefficients, Lund states the approximate equations of motion as

$$\overline{X_1}-\overline{X_2}=-M\ddot{X}_1(\alpha_{aa}+\alpha_{ab})-1\text{st mode in phase} \quad (14)$$

$$\overline{X_1}-\xi\overline{X_2}=-M\ddot{X}_1(\alpha_{aa}-\alpha_{ab})-2\text{nd mode out of phase} \quad (15)$$

Equating the rotor and the bearing forces at the bearings yields a force equation involving rotor elasticity and dimensionless bearing stiffness and damping coefficients. The corresponding equations of motion involving the support structure and bearings are also included.

In this stability analysis Lund did not attempt to solve for the log decrement of the system, but only for the threshold of stability in which the real component P of the eigenvalue λ is zero. Lund analyzes three basic cases with the rigid rotor on the gas-lubricated bearings, the flexible rotor in gas bearings, and the rigid and flexible rotor on elastic damped supports. The plots for stability of a gas bearing are complicated by the compressibility of the lubricant. Instead of a Sommerfeld number to represent the bearing, the compressibility parameter Λ is used. For constant ambient pressure P_α is used, and film viscosity Λ becomes a dimensionless speed parameter.

Figure 5 represents the stability of a rigid rotor in a gas bearing with an $L/D=0.5$. The stability is plotted as a function of critical mass (dimensionless) vs. the dimensionless speed or compressibility parameter Λ. The set of dotted lines represents the various bearing operating eccentricities and the solid lines are the lines of constant loading.

In the stability plot of Fig. 5 for the plain gas bearing, one would assume that the vertical loading is constant. For example, with a dimensionless loading of $\overline{W}=0.14$, and a dimensionless speed of $\Lambda=1$, the critical mass M is 20. As the speed increases, we move down along the constant load line. At a speed parameter of $\Lambda=20$, the critical mass is only 0.01. When actual mass is larger than the critical mass, the rotor becomes unstable. Thus we see that as speed increases for a given load, the gas bearing eventually becomes unstable.

When an instability is encountered with a gas-lubricated bearing, the occurrence may be quite sudden with dramatic consequences. Figure 6 represents the orbit of a gas bearing grinder spindle below and above the threshold of stability [7]. Figure 6 shows the dramatic, large amplitude limit cycle motion that occurred with a gas bearing rotor with only a few *RPM* difference in speed. A further increase in speed could cause bearing damage. When this effect occurs with the grinding spindle, the quality of the surface finish is extremely poor. The occurrence of this whirl phenomenon in this commercial grinder eventually caused the

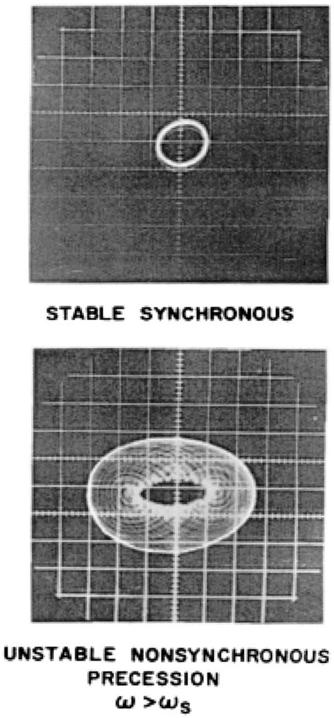

Fig. 6 Gas bearing rotor below and above the stability threshold

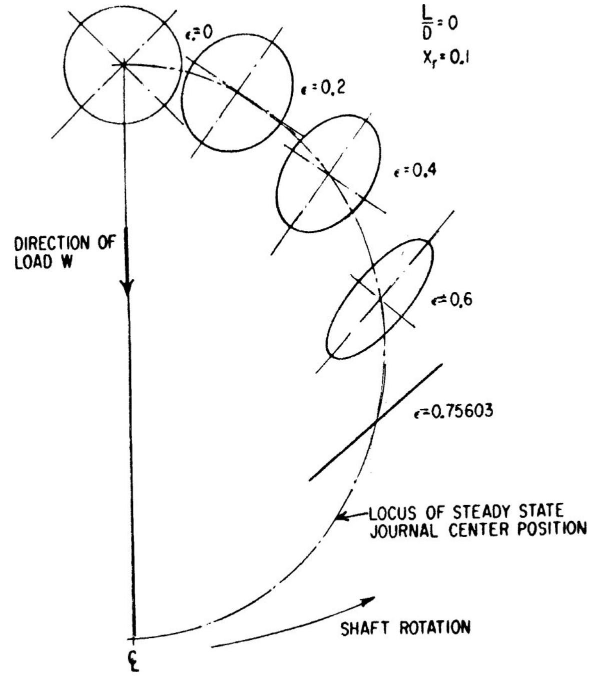

Fig. 8 Normalized whirl orbits in a sleeve bearing at the onset of instability

company to discontinue its manufacture. By the 1960's the gas bearing was also seeing wide application in dentist drills. However, the occurrence of self-excited whirling could lead to some unpleasant experiences such as the dentist removing half of a patient's tooth before he realized what had happened.

With even larger gas-bearing supported rotors, the occurrence of self-excited whirling could lead to even more unpleasant behavior. While at the Franklin Institute, we had a high speed 200 lb gas bearing rotor referred to as the EP2. (EP1 had failed before my arrival.) We had just stepped out of the lab to announce to the section chief that we had achieved 20,000 *RPM* operating speed. At that point the rotor seized, ripping out the foundation bolts and falling off the concrete superstructure. Air is not much of a boundary lubricant! Self-excited whirling of gas bearings is not to be taken lightly.

In Lund's analysis of the gas bearing on flexible damped supports, as shown in Fig. 7, he shows the dramatic effect of the combination of support flexibility and damping on the gas bearing stability threshold. For example, the dimensionless speed time of $\Lambda=10$ represents a critical mass of 0.2 for an eccentricity ratio of 0.3, as shown in Fig. 5. With a flexible support and no damping, the critical mass is reduced by an order of magnitude to 0.02. There is no change in stability for low values of damping up to 10^{-3}. After this value of damping is exceeded, the stability increases rapidly with an increase of support damping.

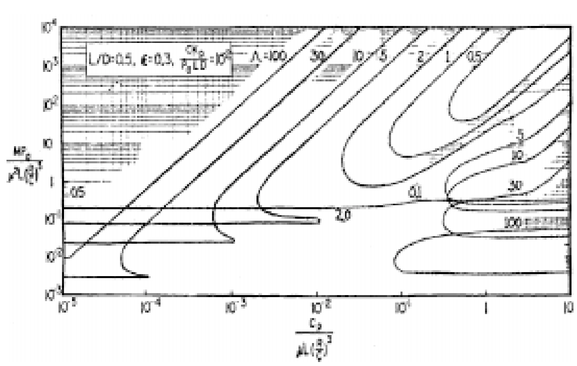

Fig. 7 Stability of cylindrical gas bearing vs. support damping for $\epsilon=0.3$, $L/D=0.5$, $\overline{K_p}=0.10$

This plot has a very practical application to the design of high speed gas bearing dental drills. By having the gas bearings supported in rubber o-rings, the stability may be greatly improved. The rubber o-rings provide both the required stiffness and damping necessary for stabilization. The limit cycle motion is also better controlled with the flexible damper supports.

Lund on Whirl Orbits in Sleeve Bearings. At the time Lund published his work on gas bearing rotors on damped flexible supports [1], he had not received his Ph.D. yet. In discussions with Jørgen, it was commented that his work would be worthy of a Ph.D. dissertation. He responded that since this work was already published, it was necessary for him to choose another topic. His Ph.D. dissertation was on the limit cycle whirl orbits of sleeve bearings and was published in the Journal of Engineering for Industry in 1967. The nonlinear equations were solved by a method of averaging, whereby the whirl orbits were obtained directly. It was found that the limit cycle whirl orbits are encountered in a narrow speed range.

Figure 8 represents the normalized whirl orbits of a sleeve bearing at the onset of oil whirl instability [8]. This figure shows the nature of the nonlinear whirl orbits at the threshold of instability. For very light loads with low bearing eccentricity rotors, the orbits are nearly circular. As the radial loading increases, the orbits become more elliptical, as seen for the case when $\epsilon=0.6$.

There are a number of important stability contributions in this paper. First, Lund shows that the bearing load vs. eccentricity for various L/D rates is best plotted using the modified Sommerfeld number σ, rather than S.

$$\sigma=\pi\left(\frac{L}{D}\right)^2 S=\frac{1}{8}\,\frac{M\omega DL}{W}\left(\frac{L}{C}\right)^2$$

For interpolation between bearing characteristics for various L/D ratios, the Lund modified Sommerfeld number is more appropriate than the conventional Sommerfeld number, S. In his analysis, Lund uses a fixed Cartesian coordinate system to express the equations of motion. This approach will become particularly important when we consider the multi-stage compressor with gyroscopic effects. Lund uses the polar coordinate system to solve for the bearing radial and tangential forces. These forces are then transformed into normalized stiffness and damping coefficients K_{ij} and C_{ij} expressed in fixed Cartesian coordinates.

Although Lund does not compute the log decrement for the linearized bearings, he does present a stability criterion for the calculation of the stability threshold based on the generalized bearing coefficients, as follows. First, the generalized stiffness K_s is computed, followed by the normalized whirl ratio ν^2.

$$K_s = \frac{K_{xx}B_{yy} + K_{yy}B_{xx} - K_{xy}B_{yx} - K_{yx}B_{xy}}{B_{xx} + B_{yy}} \tag{16}$$

$$\nu^2 = \frac{(K_{xx} - K_s)(K_{yy} - K_s) - K_{xy}K_{yx}}{B_{xx}B_{yy} - B_{xy}B_{yx}} \tag{17}$$

The stability is given by

$$\frac{CM\omega^2}{W} = \frac{\sigma K_s}{\nu^2} \tag{18}$$

The minimum value of this parameter from Fig. 5 of his paper at $\sigma = 0.2$ is about 6.4. For a horizontal rotor that is gravity loaded, in which $M = W/g$, we arrive at the simple stability criterion that

$$\omega = \sqrt{6.4}\sqrt{\frac{g}{c}} \tag{19}$$

For a 5-mil radial clearance bearing, this would place the stability threshold speed around

$$N_s RPM = 2.53 \times \frac{30}{\pi}\sqrt{\frac{386.4}{.005}} = 6{,}720 \quad RPM$$

Thus, it can be seen that it would be difficult to build a high-speed compressor mounted in sleeve film bearings to operate over 10,000 RPM!

Lund demonstrates that the equations for a flexible rotor are identical to the equations for a rigid rotor with a redefinition of the rotor mass m terms. He shows that the flexible rotor mass m_c can be calculated from

$$m_c = \frac{mK_c}{K_c + m\nu^2} \tag{20}$$

He then states that all the results for a rigid rotor can be used directly to determine the results for a flexible rotor. Thus, the more flexible the rotor is, the lower its threshold speed. In the limit, he states that a flexible rotor will go unstable at twice the rotor natural frequency and will whirl at the rotor natural frequency. Hence we see from this paper that compressor design is limited with rigid and flexible rotors to speeds of

$$\text{Rigid Rotors} - N < \frac{475}{\sqrt{C}}, \quad RPM$$

$$\text{Flexible Rotors} - N < 2N_c$$

Multi-Mass Rotor Stability by Lund

In the previous papers discussed by Lund on gas bearing and plain oil-lubricated sleeve bearings, the shaft flexibility was a simplistic model of a single K value to represent the shaft stiffness. It was mentioned that for the majority of multi-stage compressors, the rotor first critical speed and associated modal mass may be used for the assumed shaft stiffness. In the more general multi-mass analysis performed by Lund and presented in his 1974 landmark paper on stability of flexible rotors, several important additions were included which allow the theory to be applied to virtually any single span turbo-rotor.

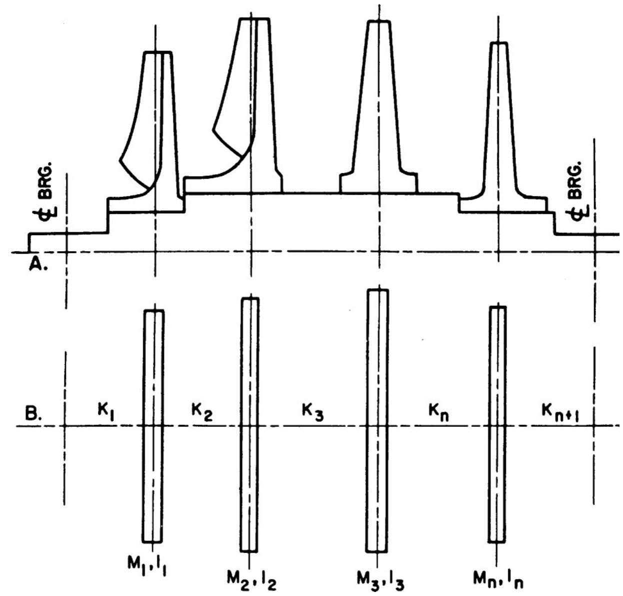

Fig. 9 Schematic diagram of a typical turbo-rotor and lumped mass representation

Figure 9 represents a schematic diagram of a multi-stage compressor on bearings. Figure 9 represents the compressor as discrete stations interconnected by springs.

In the transfer matrix analysis of Lund, he states that it is an extension of the Prohl-Myklestad method for the computation of undamped critical speeds. However, the addition by Lund to include complex motion, generalized bearings, and disk gyroscopics was a major addition. The process to search for the complex roots and obtain convergence is a problem of considerable difficulty compared to finding undamped critical speeds.

We shall address three major features of the generalized Lund flexible rotor stability analysis that allowed him to compute complex forward-backward rotor modes at any speed, including the computations of the mode log decrements. The combination of these factors created a practical design tool which is used around the world today.

Continuum Theory and Distributed Shaft Mass. A major feature of the transfer matrix method of Prohl-Myklestad [9,10] is to group the shaft parameters in one massless transfer matrix, referred to now as the field matrix, and have the mass properties, bearings, unbalance, and gyroscopic moments concentrated at a station. This matrix is referred to as the point matrix. The combination of the point and field matrices forms an element transfer matrix. Figure 10 represents the division of a shaft element by Lund into the nth and $n+1$ field and point matrices.

When using the transfer matrix method, the first question that needs to be addressed is whether to use a massless beam or base the beam transfer matrix on continuum mechanics. By the 1970's, the theory of transfer matrices by continuum mechanics was well established, as exemplified by the comprehensive text by Pestel and Leckie on "Matrix Methods in Elastic Mechanics," published in 1963 [11]. In their book they present the transfer matrices for both the massless and the distributed mass beam theory based on continuum mechanics. On initial inspection, it would appear that the continuum mechanics approach to include shaft mass would be not only preferable, but much more sophisticated and elegant.

R. L. Ruhl, in 1970, published an exceptional Ph.D. thesis on the dynamics of distributed parameter rotor systems in which he

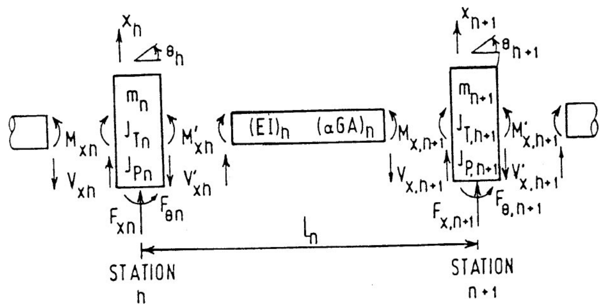

Fig. 10 Shaft element divided into point and field elements showing the sign convention

treats both the transfer matrix and finite element techniques [12]. Ruhl was ahead of his time with the finite element approach, since the required eigenvalue solvers such as *QR* and the complex Lanczos tracking-transformation eigensolver had not been perfected. The finite element rotor dynamics solution procedure today is the method of choice, but this was not a viable option in the 1970's due to insufficiency of computer storage, speed, and algorithms for solution techniques.

Ruhl, in his Ph.D. dissertation, does use the continuum approach similar to the equations of Pestel and Leckie. It turns out that this is not necessary. The continuum equations themselves are numerical disasters because of convergence problems with the hyperbolic functions. The accuracy of the solutions is not improved by continuum mechanics and results in systems which are limited in the number of mass stations that one can handle due to convergence problems.

Lund himself felt obligated to present the eigenvalues for a uniform beam on simple supports using continuum mechanics. The 4th order beam equation that Lund presents is as follows

$$\frac{\partial^2}{\partial Z^2}\left(EI\,\frac{\partial^2 Y}{\partial Z^2}\right)+\rho A\,\frac{\partial Y^2}{\partial t^2}=\delta(Z)\cdot f_o+\delta(Z-l)\cdot f_e \qquad (21)$$

Where f_o and f_e are the bearing reactions and $\delta(Z)$ is the delta function. The 4th order beam equation with flexible bearings applied at the ends leads to a transcendental equation (infinite number of solutions) and hence the computer must be used for the solution. In addition, beam representation does not include shear deformation, rotary inertia, or gyroscopic moments.

The general solution to the fourth order beam equation is of the form

$$Y(Z)=A\cos\beta Z+\beta\sin\beta Z+C\cosh\beta Z+D\sinh\beta Z \qquad (22)$$

For the uniform beam that Lund analyzes in the first part of his paper, the natural frequencies and mode shapes on stiff bearings are the classical simple support solutions.

$$Y(Z)=A_n\sin\frac{n\pi Z}{L};\quad n=1,2,3 \qquad (23)$$

$$\omega_n=(n\pi)^2\sqrt{\frac{EIg}{WL^3}},\quad \text{rad/sec} \qquad (24)$$

Figure 11 represents the mode shapes for the first two critical speeds for the uniform beam model of 50 inches in length and 4 inches in diameter with stiff supports. For the theoretical modes, the second critical will be four times the first mode.

The question with the Lund transfer matrix method is how many stations are required to accurately calculate the 1st or the 2nd mode. According to the Ruhl thesis, the finite element method is much more accurate and the transfer matrix method would require over 20 stations for reasonable accuracy. The problem with the Ruhl analysis has to do with the lumping of his mass stations. All of the stations are of equal weight.

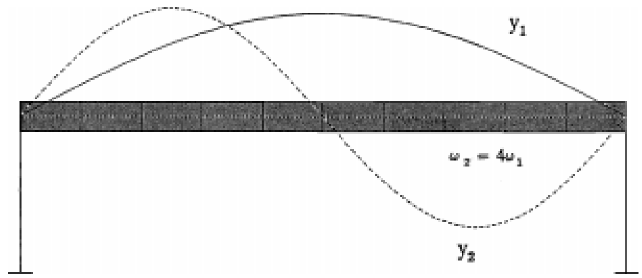

Fig. 11 First two computed modes for 11-station beam using Lund transfer matrix method

Figure 12 represents a uniform rotor represented as a three-section rotor. Using the Lund method of lumping, one-half of the weight on either side of the stations would be lumped at the stations.

By this means, the weight of w/4 would be placed at the ends. This is in comparison to the Ruhl lumping which would be w/3 at all stations. For the case of simple supports, we have reduced the uniform beam to a Jeffcott rotor similar to that shown in Fig. 2. The stiffness of the shaft with simple supports is given by

$$K=\frac{48EI}{L^3}$$

The critical speed is given by

$$\omega_c=\sqrt{\frac{96EIg}{WL^3}}=9.798\sqrt{\frac{EIg}{WL^3}} \qquad (25)$$

Whereas, the exact solution is

$$\omega_{c\ \text{exact}}=\pi^2\sqrt{\frac{EIg}{WL^3}}=9.870\sqrt{\frac{EIg}{WL^3}}$$

The percent of error between the exact and the lumped mass model is

$$\%\text{error}=\frac{9.870-9.798}{9.870}\times 100=0.73\%$$

For the higher modes, the number of mass stations required to obtain natural frequencies to less than 1% of accuracy is given by

$$N\ \text{stations}=2\times N\ \text{modes}+1 \qquad (26)$$

Thus, with the 21 stations model of Ruhl, using Lund lumping, the first 10 modes may be computed to less than 1% of error based on the Bernoulli-Euler beam assumptions. Lund, in his beam representation, also included transverse shear deformation. The im-

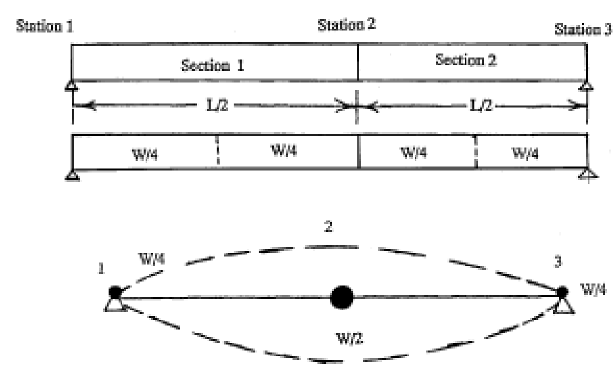

Fig. 12 Uniform beam a 3-station model showing Lund method of mass lumping

portant point is that continuum mechanics has no role in modern rotor dynamics, although many papers are presented each year using this approach. It should also be noted that in many finite element programs, such as *MSC/pal2* [13], one may choose between a lumped mass formation, a coupled mass formation, or a consistent mass formation. The lumped mass formation is identical to the Lund lumping procedure. The use of a consistent mass matrix results in natural frequencies that are slightly higher than the exact values. Therefore, there is little be gained using a finite element approach with consistent mass matrices over the Lund lumped mass model.

The representation of the rotor as a discrete system of lumped masses and massless beam elements, instead of as a continuum, is a major step in the ability to analyze a multi-mass flexible rotor. For many years this procedure was not accepted by my adviser, as well as many of the faculty in the Department of Engineering Mechanics where I obtained my degrees. I spent several years on rotor dynamic projects such as the ultra centrifuge, attempting to apply continuum mechanics to no avail.

Method of Solution

The complex roots S for the generalized eigenvalue problem of Lund are complex and the roots occur in pairs of complex conjugates. For simple undamped critical speed analysis, the roots $S_i = \pm i\omega_i$, where ω_i are the undamped critical speeds. The characteristic polynomial $P(S)$ is given by

$$P(S)=(S-S_1)(S-S_2)\ldots(S-S_n)=0 \tag{27}$$

For undamped critical speeds, it is very easy to have a numerical search along the S plane for real values of ω. This is not the situation with determining the complex roots of $P(S)$. Lund states that an eigenvalue is obtained when the boundary conditions are satisfied by

$$\Delta=\det(D)=0 \tag{28}$$

where D is the 4×4 complex system transfer matrix that generates the n th station moment and shear values from the initial displacement vector (assuming free end conditions). The root is found from a Newton-Raphson iteration scheme such as

$$S_{new}=S_{old}-\Delta_o \Bigg/ \left(\frac{d\Delta}{ds}\right)_o \tag{29}$$

The problem with this iteration scheme is that there is no way to prevent the system from converging on a previously found mode. What Lund did to make his method work was to generate a modified characteristic determinate in which the previously found root was removed from the polynomial by

$$\Delta'=\frac{\Delta}{(S-S_1)(S-S_2)} \tag{30}$$

His iteration algorithm used to avoid finding the same root repeatedly is

$$S=S_o-\Delta/d\Delta_i=d\Delta \tag{31}$$

where

$$d\Delta=\left(\frac{d\Delta}{dS_i}\right)_o-\Delta_o\sum_{j^2 1}^{J}\frac{1}{S_o-S_j} \tag{32}$$

By this procedure, the eigenvalues are determined to an accuracy of eight significant places and the number of iterations required for convergence is 5–10. Figure 13 represents the damped natural frequencies of a uniform beam with bearing stiffness values of 20,000 lb/in at each end and various values of damping.

In Fig. 13, generated by Lund, the uniform 50-in beam is supported by soft bearings of $K_b=20{,}000$ lb/in. At low damping values below $C_b=20$ lb-sec/in, the first two bearing modes are rigid body cylindrical and conical modes. The third mode is the beam free-free mode.

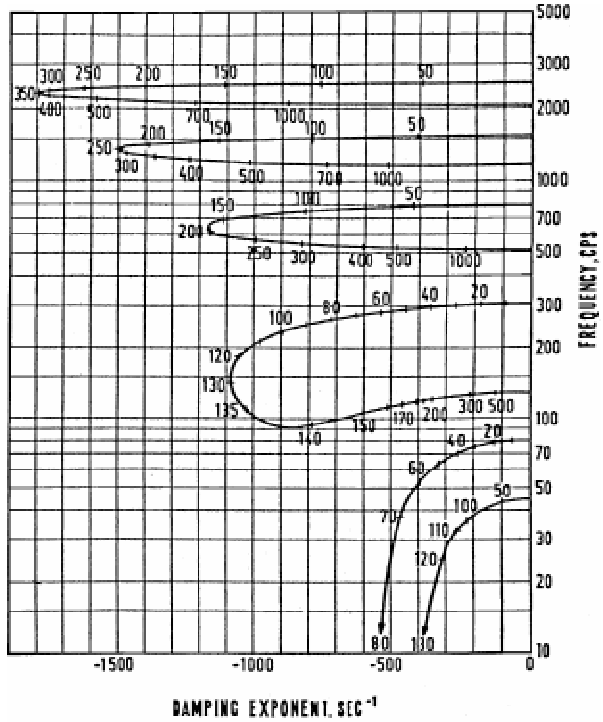

Fig. 13 Damped natural frequencies of a uniform beam with K_b=20,000 lb/in

As the bearing damping increases, the rigid body modes are critically damped as C_b increases to 138 lb-sec/in. The third mode reduces in frequency. At a bearing damping of $C_b = 130$ lb-sec/in, the maximum damping of $P=1{,}100$ rad/sec is obtained for mode 3. As the bearing damping is further increased, the effective modal damping exponent reduces. Note that at bearing damping values of $C_b>500$, the damping exponent approaches zero. This plot illustrates the concept of optimum damping for a particular mode. Above damping values of $C_b = 130$ lb-sec/in, bearing ''lockup'' begins to occur due to excessive damping. The high bearing damping causes the rotor mode shape to change from free-free to pinned. This concept of bearing lockup is very useful for the design of squeeze film dampers for compressors and turbines.

Conclusions

This paper deals specifically with Lund's contributions to the field of stability and damped eigenvalue analysis of turbomachinery. In his remarkable paper on ''Stability and Damped Critical Speeds of Flexible Rotors,'' he presents for the first time a useful numerical procedure for the calculation of compressor and turbine log decrements. By this procedure, it was now possible to compute the rotor modal amplification factors and stability threshold of compressors for various types of bearings, and also to optimize the design of squeeze film damper bearings for stabilization.

There is insufficient space to completely discuss the many contribution of Lund in rotor dynamics and fluid film bearings [14]. We could, for example, develop another section on Lund's treatment of shaft gyroscopic effects. Between his contributions to fluid film bearings and rotor dynamics, both from a stability and unbalance response standpoint, the compressor and turbine designers were now given practical design tools which helped create a new generation of high-speed, efficient turbo-rotors throughout the world.

Acknowledgments

The author wishes to acknowledge the efforts of Dr. R. Gordon Kirk for organizing the technical presentations to honor the accomplishments of J. W. Lund. I would also like to acknowledge the many personal conversations with Jørgen over the years and his personal encouragement and support.

Nomenclature

C = Bearing Damping
E = Young's Modulus
I = Shaft Moment of Inertia
K = Stiffness
L = Shaft Length
M = Mass
S = Sommerfeld Number
X = Horizontal Deflection
Y = Vertical Deflection
α = Influence Coefficient
ω = Angular Velocity
ξ = Damping Ratio
δ = Log Decrement

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