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The Effect of Support Flexibility and Damping on the Synchronous Response of a Single-Mass Flexible Rotor

This paper deals with the dynamic unbalance response and transient motion of the single-mass Jeffcott rotor in elastic bearings mounted on damped, flexible supports. A steady-state analysis of the shaft and the bearing housing motion was made by assuming synchronous precession of the system. The conditions under which the support system would act as a dynamic vibration absorber at the rotor critical speed were studied and plots of the rotor and support amplitudes, phase angles, and forces transmitted were evaluated by the computer and the performance curves were plotted by an automatic plotter unit. Curves are presented on the optimization of the support housing characteristics to attenuate the rotor synchronous unbalance response. The complete transient motion including rotor unbalance was examined by integrating the equations of motion numerically using a modified 4th-order Runge-Kutta procedure and the resulting whirl orbits were plotted by an automatic plotter unit. The results of the transient analysis are discussed with regard to the design optimization procedure derived from the steady-state analysis.

Introduction

THE STUDY of rotor dynamics has in recent years become of increasing importance in the engineering design of power systems. With the increase in performance requirements of high-speed rotating machinery in various fields such as gas turbines, process equipment, auxiliary power machinery, and space applications, the engineer is faced with the problem of designing

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a unit capable of smooth operation under various conditions of speed and load.

In many of these applications the design operating speed is often well beyond the rotor first critical speed, and under these circumstances the problem of insuring that the turbomachine will perform with a stable low-level amplitude of vibration is often difficult to achieve.

At the turn of the century Jeffcott [1]¹ developed the fundamentals of the dynamic response of the damped single-mass unbalanced rotor on a massless elastic shaft mounted on rigid bearing supports. The Jeffcott analysis of the single-mass model showed that operating speeds above the first critical speed were

¹ Numbers in brackets designate References at end of paper.

Nomenclature

A_{cr} = amplification factor at rigid support critical = $\frac{K_2}{\omega_c C_2}$ (dim)
 A_j = complex bearing amplitude, in.
 A_1 = complex support amplitude, in.
 A_2 = complex rotor amplitude, in.
 C = damping ratio = C_1/C_2 (dim)

C_b = bearing damping, lb-sec/in.
 C_c = critical damping coefficient, lb-sec/in.
 C_i = rotor internal damping, lb-sec/in.
 C_s = absolute shaft damping, lb-sec/in.
 C_1 = support damping, lb-sec/in.
 C_2 = effective rotor-bearing damping, lb-sec/in.

e_u = rotor mass eccentricity, in.
 F_1 = force transmitted to foundation, lb
 F_b = force transmitted to bearing housing, lb
 K = stiffness ratio, K_1/K_2
 K_b = bearing stiffness, lb/in.
 K_s = rotor-shaft stiffness, lb/in.
 K_1 = support stiffness, lb/in.
 K_2 = effective rotor-bearing stiffness, lb/in.

possible and that a low level of vibration would be attained once the rotor had exceeded the first critical speed.

As various compressor and turbine manufacturers adapted the flexible rotor design concept in which the rotors were designed to operate above the first critical speed, various units developed severe operating difficulties which could not be explained by the elementary Jeffcott model.

Under certain conditions of high-speed operation above the first critical speed, such influences as internal rotor friction [2], hydrodynamic bearing and seal forces [3], and aerodynamic cross-coupling [4] can lead to a destructive nonsynchronous precessive whirl motion being developed in the rotor system.

Newkirk and Kimball [5], in their early investigations of self-excited instability in compressors due to internal friction, were able to determine experimentally that the introduction of a flexible support system could greatly extend the rotor stability threshold speed. Smith [6] in 1933 was the first to verify Newkirk's findings theoretically by expanding the Jeffcott model with internal damping to include a massless damped flexible support system. Recent investigators such as Gunter [7], Tondl [8], Dimentberg [9], and others [10] have shown that flexible damped supports may improve the stability characteristics of high-speed rotors. The problem of bearing forces transmitted has been examined by various researchers [11-14]. They have shown that a significant reduction in the forces transmitted can be achieved by the proper design of the bearing support system.

The present analysis was undertaken to determine the influence of flexible supports on the synchronous unbalance response of the single-mass Jeffcott rotor, and to optimize the support system characteristics so as to minimize the rotor amplitude and forces transmitted over a given speed range. Den Hartog [15] has shown that the tuned vibration absorber will greatly reduce the response of the forced vibrations of the two-mass system. The

following analysis parallels this approach for the case of a single-mass rotor excited by an unbalance load.

This paper presents an analytic study of the tuned damper support system similar to that employed by Brock [16] and also presents a generalized study performed on the digital computer to obtain optimum support damping to produce the best response of the rotor over a wide speed range. It is well known that a damper support system can improve the vibration characteristics of a rotating shaft, and various investigators have considered the problem either from the standpoint of a continuous elastic system or as a series of lumped masses [17-23].

Although the results presented in this paper apply specifically to the single-mass Jeffcott model, the optimization procedure may be readily extended to more complex multi-mass rotor-bearing systems by employing a finite-element rotor digital computer program similar to the procedure presented by Lund [24] or by using the procedure as outlined by Crook and Grantham [25] on the vibration analysis of turbine generators on damped flexible supports.

Equations of Motion

Fig. 1 represents the single-mass Jeffcott rotor mounted in damped elastic supports. In the Jeffcott model, the shaft is considered as a massless elastic member and the rotor mass is concentrated in a disk mounted at the center of the span. The shaft is supported in linear bearings which are mounted in damped flexible supports.

Neglecting rotor acceleration and the disk gyroscopics, the governing equations of motion for the rotor, bearings, and support system in complex notation reduce to the following [31]:

$$M_2 \ddot{Z}_2 + C_s \dot{Z}_2 + C_i \dot{Z}_s - iQZ_2 + (K_s - i\omega C_i)Z_s = M_2 e_u \omega^2 e^{i\omega t} \quad (1)$$

Nomenclature

M = mass ratio, = M_1/M_2 (dim)	Z_s = complex shaft relative amplitude	$\omega_{1,2}$ = rotor system critical speeds, rad/sec
M_1 = support mass, lb-sec ² /in.	Z_1 = complex support amplitude	ω_c = rigid support critical speed, rad/sec
M_2 = rotor mass, lb-sec ² /in.	Z_2 = complex rotor amplitude, in.	Ω_1, Ω_2 = speeds at which the node points P and Q occur on response plots
N_c = rotor critical speed, rpm	Z_j = complex journal amplitude, in.	A = amplification factor at rigid support critical (dim)
P = 1st node point on response plot	α = rotor angular acceleration rad/sec ²	CB = bearing damping, lb-sec/in.
Q = rotor cross-coupling stiffness, lb/in.	β_1 = phase angle of support motion relative to rotor unbalance, deg	CD = shaft damping coefficient, lb-sec/in.
Q = 2nd node on response plots	β_2 = phase angle of rotor motion relative to rotor unbalance, deg	DC = internal damping, lb-sec/in.
R_2 = rotor absolute displacement amplitude	β_b = phase angle of bearing motion relative to rotor unbalance, deg	E = rotor mass eccentricity, in.
T = kinetic energy	θ = angular displacement, rad	FTR@WC = force transmitted at rigid support critical
TRD = transmissibility = $F_1/(M_2 e_u \omega^2)$	γ = defined as K/M (dim)	N = rotor speed, rpm
V = potential energy	ξ = damping ratio = c_1/c_c (dim)	QAC = aerodynamic cross-coupling coefficient, lb/in.
\bar{V} = velocity, in/sec	ϕ = rotor absolute amplitude phase angle, deg	TRDB = maximum bearing force transmitted (dim)
W_1 = support weight, lb	Φ = moment of inertia	TRDS = maximum support force transmitted (dim)
\bar{X} = defined as X^2	χ = optimum amplitude for tuned system	FU* = rotating unbalance load per mil unbalance eccentricity, lb
X_s = shaft relative displacement in x direction	ψ = defined as Ω_1^2 or Ω_2^2 when calculating required damping at point P or Q respectively	W = rotor speed, rad/sec
X_1 = support displacement in x direction	ω = rotor angular velocity, rad/sec	WC = rigid support critical speed, rad/sec
X_2 = rotor absolute displacement in x direction		
X_j = journal relative displacement in x direction		
Y_s = shaft relative displacement in y direction		
Y_1 = support displacement in y direction		
Y_2 = rotor absolute displacement in y direction		
Y_j = journal relative displacement in y direction		

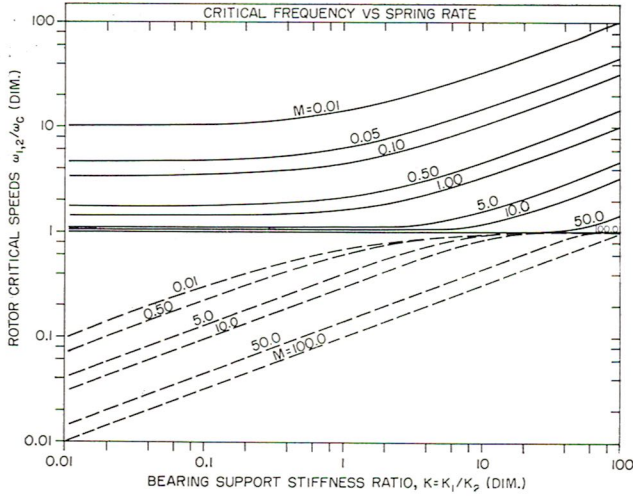


Fig. 2 Dimensionless critical speeds vs. support stiffness ratio for various support housing mass ratios

The amplification factor at the rotor critical speed is given by

$$A_{cr} = \frac{K_2}{\omega_c C_2} = \frac{250,000}{1000 \times 25} = 10.0$$

The amplification factor of 10 represents a very lightly damped rotor system and indicates that the rotor amplitude at the critical speed will be 10 times the rotor unbalance eccentricity e_u .

Rotor Response on Damped Flexible Supports

Solution of equation (6) for the case of synchronous precession for the shaft relative deflection Z_s yields

$$Z_s = (Z_2 - Z_1) \left[\frac{K_b(K_b + K_s) + (\omega C_b)^2 + i\omega C_b K_s}{(K_b + K_s)^2 + (\omega C_b)^2} \right] \quad (13)$$

Hence, in terms of the general coefficients C_2 and K_2

$$Z_s = (Z_2 - Z_1) \left[\frac{K_2 + i\omega(C_2 - C_s)}{K_s} \right] \quad (14)$$

The simultaneous equations for the absolute shaft and support housing motion reduce to the following:

$$[K_1 + K_2 - M_1\omega^2 + i\omega(C_2 + C_1 - C_s)]A_1 + [-i\omega(C_2 - C_s) - K_2]A_2 = 0 \quad (15)$$

$$[-K_2 - i\omega(C_2 - C_s)]A_1 + [K_2 - M_2\omega^2 + i\omega C_2]A_2 = M_2 e_u \omega^2 \quad (16)$$

If the damping terms are neglected, then the natural frequencies of the system may be determined by the expansion of the determinant of coefficients. The resulting frequency equation may be expressed as follows:

$$\omega_{1,2}/\omega_c = \sqrt{\frac{1}{2} + \frac{1+K}{2M}} \pm \left[\left(\frac{1+K}{2M} + \frac{1}{2} \right)^2 - \frac{K}{M} \right]^{1/2} \quad (17)$$

where

$$\omega_c = \sqrt{K_2/M_2}$$

Fig. 2 represents the dimensionless critical speeds versus the dimensionless support stiffness factor K for various values of support to rotor mass ratios. Note that the incorporation of the flexible support with the rotor-bearing system causes two critical speeds to occur, one which is higher and one which is lower than the original rotor critical on rigid supports.

To solve for the complex support and rotor amplitudes A_1 and

A_2 , equations (15) and (16) may be expressed as follows:

$$[a_{ij} + ib_{ij}]A_j = F_i; \quad j = 1, 2; \quad i = 1, 2 \quad (18)$$

Multiplying equation (18) by the complex inverse matrix of coefficients and expanding yields

$$A_1 = \frac{\begin{vmatrix} F_1 & a_{12} + ib_{12} \\ F_2 & a_{22} + ib_{22} \end{vmatrix}}{\Delta} \quad (19)$$

where

$$\Delta = d_r + id_i$$

$$d_r = (K_2 - M_2\omega^2)(K_1 - M_1\omega^2) - K_2 M_2 \omega^2 - C_1 C_2 \omega^2 - \omega^2 C_s (C_2 - C_s)$$

$$d_i = C_1 \omega (K_2 - M_2\omega^2) + C_2 \omega (K_1 - M_1\omega^2 - M_2\omega^2) + C_s \omega (K_2 + M_2\omega^2)$$

Expanding equation (19)

$$A_1 = \frac{F_1 a_{22} - F_2 a_{12} + i(F_1 b_{22} - F_2 b_{12})}{d_r + id_i} \quad (20)$$

In this case only an external unbalance excitation force F_2 is acting on the shaft and no external exciting force F_1 is assumed to be present on the support system. For example, an excitation force F_1 may be transmitted to the rotor system through the support structure by vibrations of auxiliary or adjacent equipment.

$$A_1 = - \frac{F_2 [a_{12} d_r + b_{12} d_i + i(b_{12} d_r - a_{12} d_i)]}{d_r^2 + d_i^2} \quad (21)$$

Assume A_1 is of the form

$$A_1 = A_{1r} - iA_{1i} \quad (22)$$

The complex support amplitude Z_1 after some complex algebraic manipulation is given by

$$Z_1 = A_1 e^{i\omega t} = R_1 e^{i(\omega t - \beta_1)} \quad (23)$$

where

$$R_1 = \sqrt{A_{1r}^2 + A_{1i}^2}, \quad \beta_1 = \tan^{-1} \left(\frac{d_i}{d_r} \right)$$

If the shaft damping coefficient C_s is considered small in comparison to the effective damping coefficient C_2 then the system displacements and phase angles are given as follows:

$$R_1 = M_2 e_u \omega^2 \sqrt{\frac{K_2^2 + (\omega C_2)^2}{d_r^2 + d_i^2}} \quad (24)$$

and the phase angle of the support motion relative to the rotating unbalance load is given by

$$\beta_1 = \tan^{-1} \left[\frac{K_2 d_i - \omega C_2 d_r}{K_2 d_r + \omega C_2 d_i} \right] \quad (25)$$

since the complex rotor support motion Z_1 is given by

$$Z_1 = X_1 + iY_1 = R_1 e^{i(\omega t - \beta_1)}$$

Then, for example, the horizontal and vertical components of the support motion are given by

$$\begin{cases} X_1 \\ Y_1 \end{cases} = M_2 e_u \omega^2 \sqrt{\frac{K_2^2 + (\omega C_2)^2}{d_r^2 + d_i^2}} \begin{cases} \cos(\omega t - \beta_1) \\ \sin(\omega t - \beta_1) \end{cases} \quad (26)$$

In a similar fashion, the complex rotor amplitude Z_2 is given by

$$Z_2 = M_2 e_u \omega^2 \frac{a_{11} + ib_{11}}{d_r + id_i} e^{i\omega t} \quad (27)$$

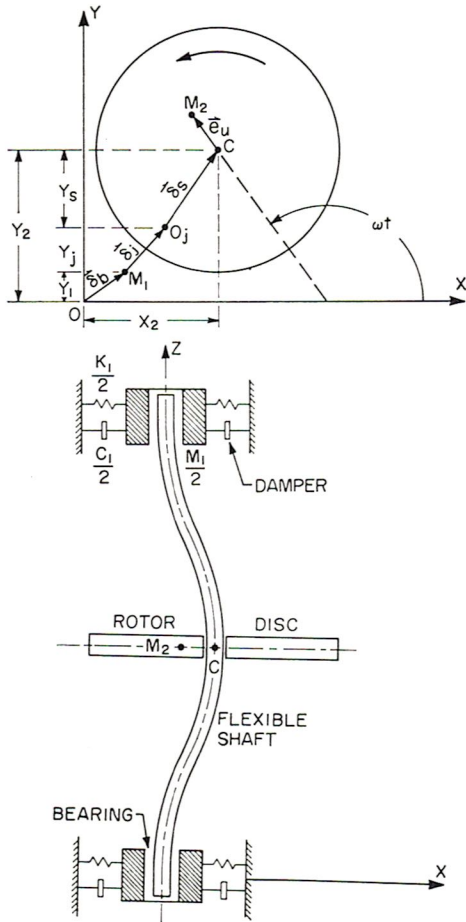


Fig. 1 Schematic diagram of single-mass rotor on damped elastic supports

$$C_i \dot{Z}_j - C_i \dot{Z}_s + K_b Z_j - (K_s - i\omega C_i) Z_s = 0 \quad (2)$$

$$M_1 \ddot{Z}_1 + C_1 \dot{Z}_1 + K_1 Z_1 - C_i \dot{Z}_s - (K_s - i\omega C_i) Z_s = 0 \quad (3)$$

where

$$Z_s = Z_2 - Z_j - Z_1 = \text{relative shaft deflection}$$

If the internal damping C_i and the aerodynamic cross-coupling term Q are excluded from the above equations then the system will be stable [26].

After the initial transient motion has damped out, it may be assumed that the system steady-state motion is circular synchronous precession. In this case the displacements are related to the velocity and acceleration vectors as follows:

$$\begin{aligned} Z_i &= A_i e^{i\omega t} \\ \dot{Z}_i &= i\omega Z_i \\ \ddot{Z}_i &= i\omega \dot{Z}_i = -\omega^2 Z_i \end{aligned} \quad (4)$$

where A_i is in general complex.

The differential equations of motion may be reduced to a set of algebraic equations for the determination of the rotor steady-state motion.

$$(K_s - M_2\omega^2 + iC_s\omega)A_2 - K_s A_j - K_s A_1 = M_2 e_u \omega^2 \quad (5)$$

$$-K_s A_2 + (K_b + K_s + i\omega C_b)A_j + K_s A_1 = 0 \quad (6)$$

$$-K_s A_2 + K_s A_j + (K_1 + K_s - M_1\omega^2 + i\omega C_1)A_1 = 0 \quad (7)$$

Rotor Amplification Factor

Consider the steady-state orbit of the flexible rotor on rigid

supports. The rotor amplitude is a function of both the rotor and bearing stiffness and damping characteristics. Assuming A_1 is zero, the relative journal bearing complex amplitude from equation (6) is given by

$$A_j = \frac{K_s(K_s + K_b - i\omega C_b)}{(K_s + K_b)^2 + (\omega C_b)^2} A_2 \quad (8)$$

Solving equation (5) for the rotor amplitude yields

$$A_2 = M_2 e_u \omega^2 \frac{(K_1 - M_2\omega^2 - i\omega C_2)}{(K_2 - M_2\omega^2)^2 + (\omega C_2)^2} \quad (9)$$

where

$$K_2 = \frac{K_b K_s (K_s + K_b) + K_s (\omega C_b)^2}{(K_s + K_b)^2 + (\omega C_b)^2}$$

$$C_2 = \frac{K_s^2 C_b}{(K_b + K_s)^2 + (\omega C_b)^2} + C_s$$

The rotor displacement vector Z_2 may be expressed in terms of the absolute displacement R_2 and the phase angle ϕ as follows:

$$Z_2 = R_2 e^{i(\omega t - \phi)} \quad (10)$$

where

$$R_2 = \frac{M_2 e_u \omega^2}{\sqrt{(K_2 - M_2\omega^2)^2 + (\omega C_2)^2}}$$

$$\phi = \tan^{-1} \left[\frac{\omega C_2}{K_2 - M_2\omega^2} \right]$$

The above results are similar to the rotor amplitude and phase angle results for the single-mass flexible rotor on rigid supports as shown by Thomson [27].

The rotor undamped or natural critical speed is given by

$$\omega_c = \sqrt{\frac{K_2}{M_2}} = \sqrt{\frac{K_b K_s}{(K_b + K_s) M_2}} \quad (11)$$

For the case of a lightly damped rotor system on rigid supports the maximum rotor amplitude will occur at approximately the rotor critical speed and the dimensionless rotor amplitude or amplification factor at the critical speed is given by

$$A = \left. \frac{R_2}{e_u} \right|_{\omega = \omega_c} = \frac{K_2}{\omega_c C_2} \quad (12)$$

Example 1. Consider a 97-lb disk centered on a uniform massless elastic shaft as shown in Fig. 1. Assume that the bearing stiffness $K_b/2$ is 500,000 lb/in. and that the effective shaft stiffness K_s at the disk station is 333,000 lb/in. Assuming light damping, the total stiffness K_2 is given by

$$K_2 = \frac{K_s K_b}{K_s + K_b} = \frac{1 \times 0.333 \times 10^{12}}{(1 + 0.333)10^6} = 250,000 \text{ lb/in.}$$

The rotor critical speed is

$$\omega_c = \sqrt{\frac{K_2}{M_2}} = \sqrt{\frac{250,000}{0.25}} = 1000 \text{ rad/sec}$$

or $N_c = 9550$ rpm.

If the rotor damping C_s is assumed to be 15 lb-sec/in. and the bearing damping coefficient $C_b/2$ is 80 lb-sec/in., then the effective system damping coefficient C_2 is approximately given by

$$\begin{aligned} C_2 &\approx C_s + \frac{K_s^2 C_b}{(K_b + K_s)^2} \\ &= 15 + \frac{(0.333)^2 \times 10^{12} \times 160}{(1.333)^2 \times 10^{12}} = 25 \text{ lb-sec/in.} \end{aligned}$$

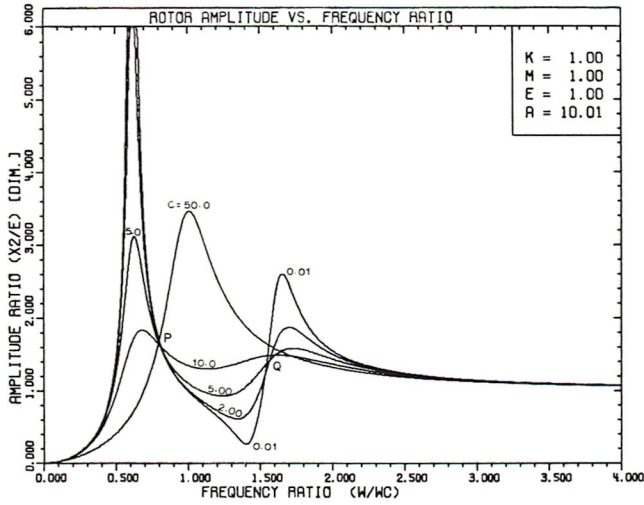


Fig. 4 Absolute rotor motion with a tuned support system for various values of support damping. $K = M = 1$, $A = 10$

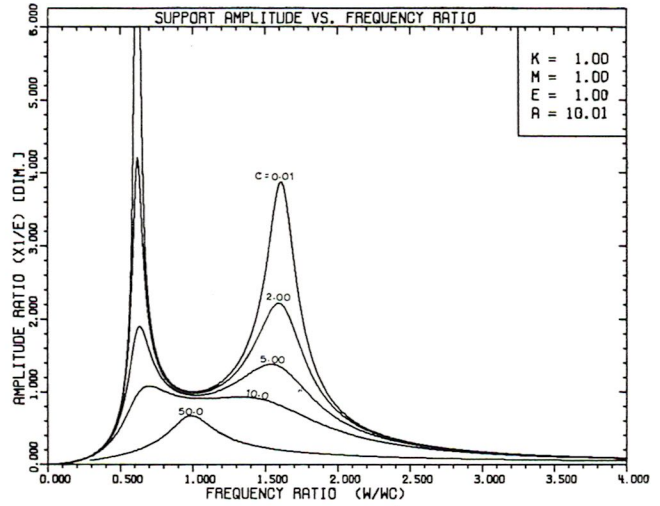


Fig. 6 Support amplitude vs. speed for various values of support damping

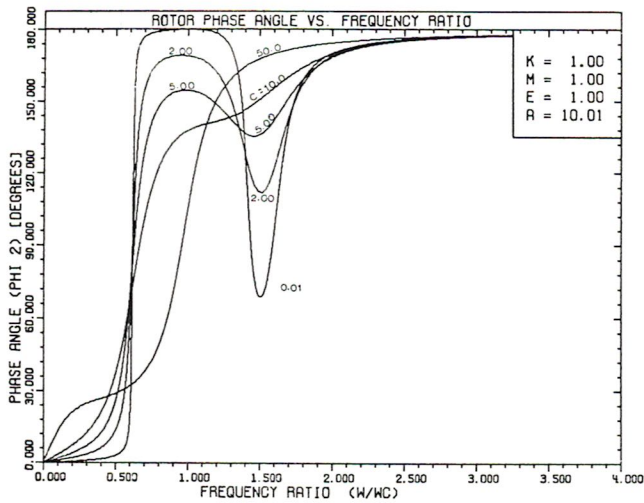


Fig. 5 Phase angle of absolute rotor motion relative to unbalance for various values of support damping

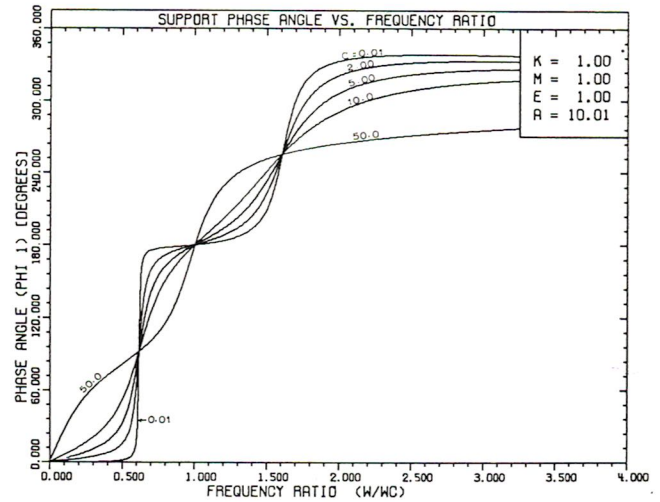


Fig. 7 Phase angle of support motion relative to rotor unbalance for various values of support damping

As the support damping approaches infinity, the rotor amplitude will asymptotically approach 10.

Fig. 4 represents the absolute dimensionless rotor motion for various values of support damping ratio and is similar to Fig. 3. It should be noted that the damping coefficient of 10 also appears to be close to the optimum damping for the absolute motion as well as the relative motion.

It is of interest to note that the various damping lines all intersect at a common point *P* in the plot of absolute as well as relative rotor motion. If the rotor amplification factor *A* is 100 (implying light rotor damping) then there will be two common points of intersection *P* and *Q* on the response plots (see Fig. 10) similar to that shown by Den Hartog for the damped vibration absorber [15]. The intersection points *P* and *Q* will occur at speeds respectively below and above the rigid support critical speed. The rotor amplitude may be minimized for the case of the absolute rotor motion by selecting the damping such that the slope of the response curve is zero at point *P*, and zero at point *Q* to minimize the rotor relative motion.

Fig. 5 represents the phase angle between the rotating unbalance vector and the absolute rotor displacement vector for various damping coefficients. The phase angle for the single-mass rotor on rigid supports (Jeffcott model) increases with speed from 0 to 90 deg at the critical speed and asymptotically

approaches 180 deg as the rotor speed greatly exceeds the critical speed. The phase angles of the rotor on damped flexible supports has a considerably different behavior from that of the rigid support rotor. For light values of support damping ($C = 0.01$), the phase angle increases rapidly to 180 deg as the system passes through the first critical speed and drops to almost 60 deg as it passes through the second critical speed. As the speed greatly exceeds the highest critical speed, the phase angle again approaches 180 deg. The phase angle of 180 deg indicates that the rotor mass center lies along the rotor spin axis. As the support damping coefficient is increased beyond 5 for the case of the tuned system, the reduction in phase angle above the first critical speed is suppressed. This phenomena of phase angle reversal above the first critical speed has been observed experimentally [30].

Fig. 6 represents the support amplitude versus speed for various damping values and indicates that with very light support damping there will be large support resonances. As the damping is increased beyond $C = 10$ the resonances are suppressed and the amplitude is only slightly greater than 1. For $C = 50$ there is only a small peak observed in the support system which occurs at a speed corresponding to the rigid support critical speed. The addition of high damping ($C > 50$) freezes the support and limits its motion drastically.

After some manipulation, equation (27) reduces to the following:

$$Z_2 = M_2 e_u \omega^2 \sqrt{\frac{(K_1 + K_2 - M_1 \omega^2)^2 + [(C_1 + C_2)\omega]^2}{d_r^2 + d_i^2}} e^{i(\omega t - \beta_2)} \quad (28)$$

where

$$\beta_2 = \tan^{-1} \left(\frac{(K_1 + K_2 - M_1 \omega^2)d_i - (C_1 + C_2)\omega d_r}{(K_1 + K_2 - M_1 \omega^2)d_r + (C_1 + C_2)\omega d_i} \right) \quad (29)$$

The relative journal displacement is given by

$$Z_j = Z_2 - Z_1 - Z_s \quad (30)$$

Where the relative shaft deflection is

$$Z_s = \frac{(Z_2 - Z_1)}{K_s} [K_s - K_2 - i\omega C_2] \quad (31)$$

Solving for the journal displacement

$$Z_j = R_j e^{i(\omega t - \beta_j)}$$

where

$$R_j = M_2 e_u \omega^2 \times \sqrt{\left[\frac{(K_1 - M_1 \omega^2)^2 + (\omega C_1)^2}{d_r^2 + d_i^2} \right]} \times \sqrt{\left[\left(\frac{K_s - K_2}{K_s} \right)^2 + \left(\frac{\omega C_2}{K_s} \right)^2 \right]} \quad (32)$$

and the phase angle β_j between the journal amplitude and rotating unbalance force is given by

$$\beta_j = \tan^{-1} \left(\frac{(K_1 - M_1 \omega^2)d_i - \omega C_1 d_r}{(K_1 - M_1 \omega^2)d_i + \omega C_1 d_r} \right) + \tan^{-1} \left(\frac{\omega C_2}{K_s - K_2} \right) \quad (33)$$

Forces Transmitted

The magnitude of the resultant forces transmitted through the bearings and the support are of considerable interest to the designer from a standpoint of bearing life and system isolation. It is desirable to minimize the forces transmitted through the supporting structure and foundation so that other machines or piping systems are not excited. The magnitude of the force transmitted through the bearings is given by

$$F_b = R_j \sqrt{K_b^2 + (\omega C_b)^2} \quad (34)$$

and the force transmitted through the support system is given by

$$F_1 = R_1 \sqrt{K_1^2 + (\omega C_1)^2} \quad (35)$$

An indication of the effectiveness of the support system in attenuating the forces transmitted to the foundation is the support dynamic transmissibility factor TRD which will be defined as the ratio of the magnitude of the transmitted support force to the rotating unbalance load. If the dynamic transmissibility is less than 1, then the support system possesses good attenuation characteristics. Analysis has shown that if the support housing impedance characteristics, which are determined by the housing mass, stiffness, and damping, are mismatched to the rotor-bearing system then under certain speed conditions the dynamic transmissibility may exceed 1.

The dynamic transmissibility for the support is defined as

$$TRD = \frac{F_1}{M_2 e_u \omega^2} = \sqrt{\frac{[K_2^2 + (\omega C_2)^2][K_1^2 + (C_1 \omega)^2]}{d_r^2 + d_i^2}} \quad (36)$$

If it is assumed that the rotor is operating well above any of the system critical speeds then the dynamic transmissibility is approximately given by

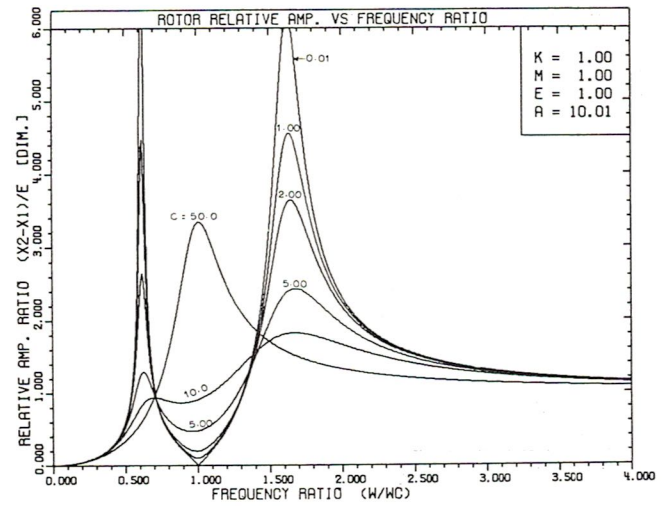


Fig. 3 Dimensionless relative rotor amplitude vs. speed ratio for various values of support damping for a tuned support system, $K = M = 1$

$$TRD \approx \frac{1}{\omega^4} \sqrt{\frac{[K_2^2 + (\omega C_2)^2][K_1^2 + (C_1 \omega)^2]}{M_1^2 M_2^2}} \quad (37)$$

The above expression leads to the well-known conclusion that to minimize the forces transmitted through the support for supercritical speed operation in the Jeffcott model, the support damping should be zero and the support stiffness should be as light as possible [28]. This is a highly undesirable design practice for several reasons since large rotor amplitudes and forces transmitted may be encountered when passing through the rotor critical speeds, and also the rotor system would be extremely shock-sensitive and particularly susceptible to self-excited whirl instability under such conditions.

A compromise support damping coefficient should be selected to either minimize the rotor amplitudes or the forces transmitted over the operating speed range and also be sufficient to insure adequate rotor stability.

Analysis of System Unbalance Response—Tuned System

Fig. 3 represents a computer-generated plot of the dimensionless rotor relative amplitude versus the dimensionless rotor speed for the case of $K = M = 1$. This relative rotor amplitude is equivalent to the motion monitored by a proximity probe mounted in the casing measuring the rotor motion at the center span. This system represents a tuned condition in which the support stiffness ratio K is equal to the support mass ratio M . With no support damping in the system, the tuned support will cause the relative rotor amplitude to be zero at a speed corresponding to the rotor critical speed with rigid supports. The introduction of support mass and flexibility has caused two critical speeds to appear in the system, one above and one below the rigid support rotor critical. Note that when the support damping is relatively low the amplitudes at the two criticals becomes extremely high.

As the dimensionless support damping ratio C increases from 0.01 to 10 the rotor amplitudes at the system critical speeds decrease while the amplitude increases at a speed corresponding to the rigid support critical speed ($\omega/\omega_c = 1$). Note that in this case the damping value of 10 appears to be close to an optimum value for the minimization of the resonance amplitudes. If the support damping is further increased from 10 to 50, Fig. 3 indicates that there will be only one critical speed present in the system which will correspond to the rigid support critical. Although the damping of $C = 50$ is over 5 times the optimum value, the maximum amplitude is only $1/3$ the rigid support value of 10.

$$\xi^2 = \frac{4M^2\psi^3 - 3M^2(4M+3)\psi^2 + M(12M^2+13M+8)\psi - M(1+2M)^2}{-12M\psi^2 + 8(1+2M)\psi - (1+2M)} \quad (39)$$

where

$$\xi = C_1/C_c = C_1/C_2 \times (1/2A) = C_1 \frac{\omega_c}{2K_2}$$

$\psi = \Omega_1^2$ or Ω_2^2 depending on whether the value calculated is for point P or Q respectively

and

$$\Omega_1^2 = \frac{\sqrt{1+2M}}{1+\sqrt{1+2M}}$$

$$\Omega_2^2 = \frac{\sqrt{1+2M}}{\sqrt{1+2M}-1}$$

For example, when $M = 1$ and for the first node, P ,

$$\psi = \Omega_1^2 = \frac{\sqrt{3}}{1+\sqrt{3}} = 0.634$$

and

$$\xi^2 = 0.447$$

Hence

$$\left. \frac{C_1}{C_c} \right|_{\text{opt}} = 0.688 \quad \text{for point } P$$

In a similar fashion

$$\left. \frac{C_1}{C_c} \right|_{\text{opt}} = 0.559 \quad \text{for point } Q$$

Example 2. As an example of the application of the tuned support design criteria consider the rotor of Example 1 mounted in flexibly supported bearing housings which weigh 48.5 lb and have a stiffness of 125,000 lb/in. The total support weight W_1 and stiffness K_1 are given by

$$W_1 = 2 \times 48.5 = 97 \text{ lb}$$

$$K_1 = 2 \times 125,000 = 250,000 \text{ lb/in.}$$

Hence

$$M = M_1/M_2 = 1.0$$

$$K = K_1/K_2 = 1.0$$

The critical damping coefficient C_c is given by

$$C_c = \frac{2K_2}{\omega_c} = \frac{500,000 \text{ lb/in.}}{1000 \text{ rad/sec}} = 500 \text{ lb-sec/in.}$$

Thus the support damping coefficients required to make the slope of the rotor amplitude curve zero at points P and Q are respectively given as follows

$$C_1|_p = 0.688 \times C_c = 344 \text{ lb-sec/in.}$$

$$C_1|_q = 0.559 \times C_c = 279.5 \text{ lb-sec/in.}$$

These calculations are valid only for the case of zero damping on the rotor and in the bearings (i.e., $A = \infty$) and only for the tuned system (i.e., $K = M$). For a more realistic solution, a value of $A = 10$ was chosen and numerous cases were then programmed on a digital computer to arrive at a value of optimum amplitude and required damping. This approach is discussed in the next section of this paper but the results for the tuned system are very nearly the same as the results arrived at analytically for the case of $A = \infty$ and are presented in Fig. 11.

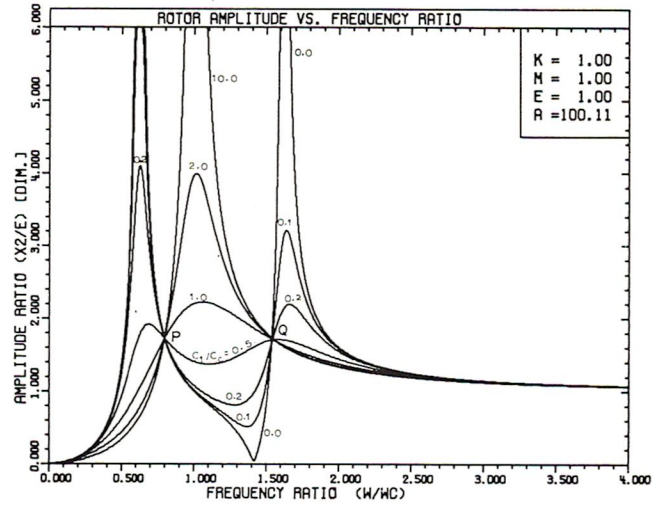


Fig. 10 Amplitude of motion vs. speed with light rotor damping ($A = 100$) for various values of support damping, $K = M = 1$

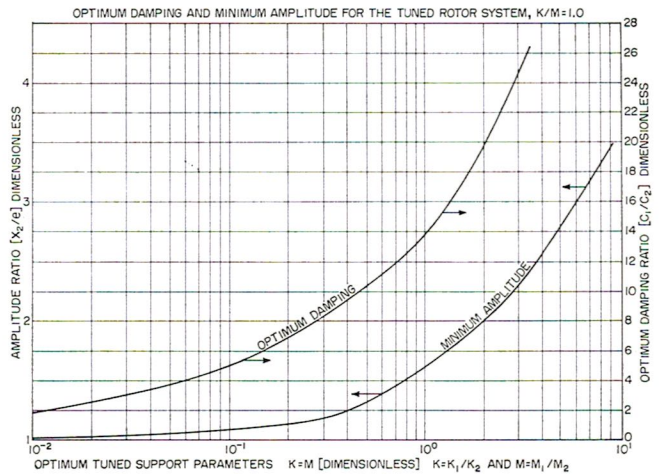


Fig. 11 Optimum support damping and maximum rotor amplitude vs. mass ratio for $A = 10$

The results shown in Fig. 11 are approximately correct for systems having moderate to light damping on the rotor (i.e., $10 \leq A < \infty$). Note that the smaller the mass ratio M , the lower will be the peak response and also the lower will be the required support damping. For example, if the mass ratio is 0.1, then the maximum dimensionless amplitude will be only 1.1 and the required damping ratio will be 5 as compared to a value of 13.6 for an M ratio of 1. Fig. 12 is a response plot for the tuned system $K = M = 0.1$ which illustrates the validity of the results plotted in Fig. 11. The response curve for a damping ratio of 5 passes almost horizontal through the node point and has the low amplitude ratio as indicated by Fig. 11.

Example 3. Consider a rotor system similar to Example 2 in which the rotor rigid support amplification factor $A = 10$.

For a tuned support system the dimensionless support damping coefficient is obtained from Fig. 11 for $M = 1$ as follows:

$$C = C_1/C_2 = 13.6$$

where C_2 is given as 25 lb-sec/in. (Example 1).

Therefore,

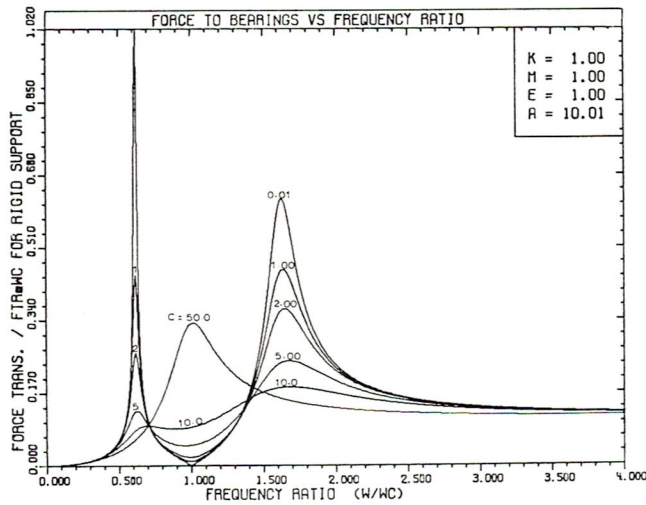


Fig. 8 Dimensionless force transmitted to bearings vs. speed ratio for various values of support damping

Fig. 7 represents the support phase angles versus speed ratio for various values of support damping. The phase angle for light damping ($C = 0.01$) is zero at low speeds and goes to 180 deg as it passes through the first critical and then shifts to 330 deg upon passing through the second critical speed. If the rotor damping is light ($A = 100$) the support phase angle will approach 360 deg after passing through the second critical speed. Note that the various damping lines intersect at three points. The first node point represents the first system critical speed, the second node point represents the rigid support critical speed, and the third node point represents the second critical speed on flexible supports. In the discussion of the single-mass flexible rotor presented in vibration texts [27] the phase change is only shown from 0 to 180 deg. In more complex systems with flexible supports, the phase change may vary between 0 and 360 deg. For example in multi-mass systems the authors have observed phase changes of n times 180 deg where n represents the number of system critical speeds. The measurement of rotor and support phase angles have been neglected and limited data has been reported in the literature. This is an extremely useful variable which when incorporated with displacement measurements can be used in balancing flexible rotors or impedance calculations of the support system.

Fig. 8 represents the dimensionless bearing forces transmitted for the tuned system. The dimensionless force transmitted is obtained by dividing by the transmitted force corresponding to the value at the critical speed of the original rotor on rigid supports. Because of the light shaft damping the force transmitted curves are similar in appearance to the displacement curves. Note that for the support damping coefficient of $C = 10$ the forces transmitted to the bearings at the rigid support critical are only 10 percent of the value transmitted for the rotor bearing system on rigid supports.

Fig. 9 represents the force transmitted through the bearing supports to the foundation or base for various values of support damping. With a very lightly damped support system ($C = 0.01$) the support amplitude and force transmitted will be particularly high at the first critical speed where the bearing and support motions are in phase. At the second critical speed, the support amplitude is lower than the amplitude attained at the first critical speed. This is because the bearings and support motions are out of phase which enables the bearing damping to help attenuate the support motion. It is of interest to note from Fig. 8, for the tuned rotor system, the bearing force transmitted at $(\omega/\omega_c) = 1$ with an undamped support system is 0. Fig. 9 shows that the corresponding force transmitted through the sup-

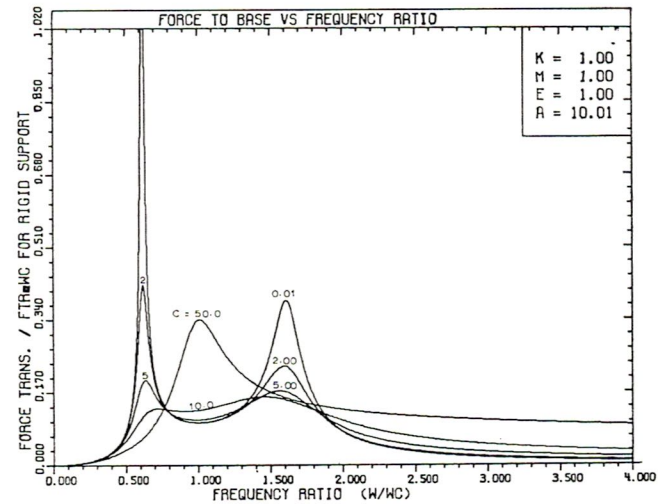


Fig. 9 Dimensionless force transmitted to foundation vs. speed ratio for various values of support damping

port at $\omega/\omega_c = 1$ has been reduced to only 10 percent of the rigid support value.

The force transmitted for an undamped support system at a speed ratio of four is approximately 10 percent of the rigid support value. This condition would be desirable if it were possible to accelerate through the criticals, thereby avoiding the large steady-state amplitudes and forces developed.

The near optimum damping of 10 increases the support forces transmitted in the supercritical region to 30 percent of the rigid support value and the overdamped support system ($C = 50$) has increased to nearly 80 percent. Hence, the support damping introduced to suppress the system resonances will cause the forces transmitted to increase in the supercritical speed region.

If the system is designed to operate over the entire speed range shown, then the near-optimum value of damping (i.e., $C = 10$) for suppressing the rotor absolute amplitude also produces the most desirable attenuation of forces to the system support structure.

Optimum Damping for Tuned System

From the observation of the computer-generated displacement and force transmitted plots it is apparent that there exists an optimum damping to either minimize the rotor amplitudes or the forces transmitted over the entire speed range.

For example, to minimize the absolute rotor motion as shown in Fig. 4 or the relative rotor motion shown in Fig. 3, the method of [16] may be used in which the damping is chosen so that the slope of the amplitude curve is zero at points P and Q respectively. In the tuned system where $K/M = 1$ for light rotor damping ($A = 100$), the rotor amplitudes at points P and Q are independent of the support damping as shown in Fig. 10 and can be shown to be equal to

$$X_2 = x_2/e_u|_{P,Q} = \sqrt{1 + 2M} \quad (38)$$

Therefore with the tuned system illustrated with a mass ratio of $M = 1$, the maximum amplitude at P or Q will be 1.732 times the rotor unbalance eccentricity. The optimum damping may be selected so that the tangent to the amplitude curve at either point P or Q has a zero slope. By selecting the optimum damping in this fashion it is seen that the maximum amplitude in the system will not exceed the value given by equation (38). Thus it is readily apparent that to minimize the rotor response over a given speed range, the support mass should be kept as light as possible.

After considerable algebraic manipulation [28] the optimum damping coefficient for both points P and Q is given by the following expression:

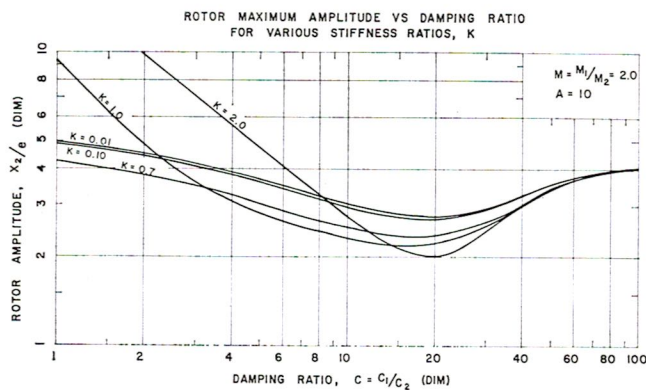


Fig. 15 Rotor maximum amplitude vs. damping ratio for various values of stiffness ratios for a high mass ratio support, $M = 2$, $A = 10$

It is also of interest to note that if a high support stiffness ($K = 2$) is used in conjunction with a low value of support damping ($C < 2$) then the rotor response will be worse than the original rotor response on rigid supports ($A = 10$).

Fig. 15 represents the maximum rotor response versus support damping for a high support mass ratio system ($M = 2$). It is obvious from the comparison of Figs. 14 and 15 that the high mass ratio support system is less desirable. The minimum rotor amplitude that can be achieved is $x_2/e_u = 2$ with a tuned support where $K = M = 2$ and a support damping coefficient of $C = 20$. (Also see Fig. 11 on the tuned system.) As the support stiffness ratio is reduced, the rotor response curve increases in the optimum damping region.

If it is not possible to incorporate a high value of support damping into the system ($C = 20$), then the rotor amplitude can still be reduced to 40 percent of the original rotor response by a low support damping value of $C = 1$ and a reduced support stiffness ratio of $K = 0.7$. For low values of support damping, if the support stiffness increases beyond $K = 0.7$, the rotor response rapidly increases.

A series of plots similar to Figs. 14 and 15 was produced for various mass ratios in order to determine the optimum rotor response for off-tuned support conditions. Fig. 16 represents the rotor maximum amplitude versus the support mass ratio for various values of support stiffness with optimum damping.

For the case of $A = 10$, Fig. 16 illustrates that the lowest amplitude can be achieved with a low mass ratio support system. With a high mass ratio support system such as $M = 5$, the rotor amplitude X_2 can be reduced from 10 to 2.8 by means of a tuned support stiffness of $K = 5.0$ and optimum damping. Note that as the support stiffness becomes very light, the maximum rotor amplitude increases to 7.5.

At a low value of support mass ($M = 0.1$), the rotor amplitude increases as the support stiffness increases. The optimum damping required with the tuned support is given by the following approximate relationship:

$$c_1 = \frac{1.37 \times K_2}{\omega_c} \times M^{0.437} \quad (40)$$

Transient Analysis

The previous discussion has been concerned only with the steady-state response of the rotor due to unbalance and has not considered the rotor initial transient motion. As discussed previously, the damped flexible support system is important, not only from the standpoint of reduction of synchronous unbalance response, but also in the control of self-excited vibrations such as caused by internal friction, aerodynamic excitation, etc. Therefore, to investigate the general rotor motion and also to provide a check on the steady-state analysis, the rotor equations of motion were integrated forward in time on the digital computer using a

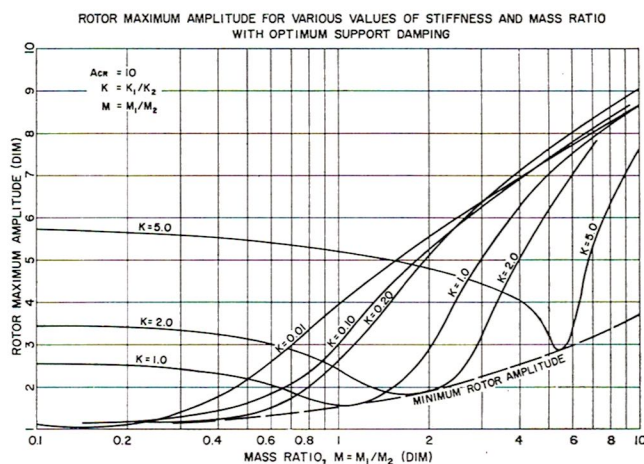


Fig. 16 Rotor maximum amplitude for various values of stiffness and mass ratio with optimum support damping

ABSOLUTE ROTOR MOTION

$N = 30000 \text{ RPM}$	$M = 0.100$
$K = 0.092$	$C = 43.946$
$W_2 = 96.60 \text{ LB.}$	$K_B = 500,000 \text{ LB/IN}$
$K_S = 500,000 \text{ LB/IN}$	$C_B = 100.0 \text{ LB-SEC/IN}$
$D_C = 0.5 \text{ LB-SEC/IN}$	$W_1 = 9.66 \text{ LB.}$
$D_D = 0.5 \text{ LB-SEC/IN}$	$K_1 = 25,000 \text{ LB/IN}$
$Q_A C = 0.5 \text{ LB/IN}$	$C_1 = 1000.0 \text{ LB-SEC/IN}$
$T_{A D B} = 0.585 \text{ AND OCCURS AT } 0.58 \text{ CYCLES}$	
$T_{A D S} = 2.159 \text{ AND OCCURS AT } 0.99 \text{ CYCLES}$	
$F_U = 2469.026 \text{ LBS.}$	

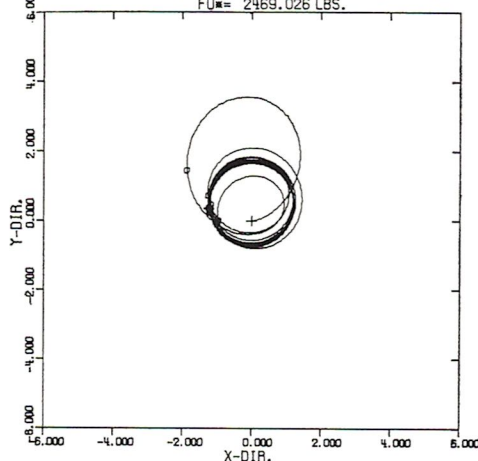


Fig. 17 Dimensionless transient motion of an unbalanced rotor for 12 cycles on overdamped supports, $K = M = 0.1$, $C = 44$

modified 4th-order Runge-Kutta integration procedure. This procedure is of importance particularly if the analysis is extended from a linear bearing or support system to include a nonlinear hydrodynamic damper bearing as presented in [13].

The dimensionless rotor and support transient orbits were automatically computer-plotted with the following dimensionless parameters:

$$X = x/e_u, \quad Y = y/e_u$$

Fig. 17 represents the initial transient orbit of a 96.6-lb rotor of Example 1 with a highly damped support ($C = 43$) for the first 12 cycles of shaft motion. The support mass ratio and the support stiffness ratio are both approximately the same (0.10) which represents a tuned system. Because of the excessive support damping, the maximum force transmitted to the support is 2.16 times the unbalance force while the force transmitted to the bearings is reduced by about 40 percent. The magnifications of the force to the support would be highly undesirable for applications such as aircraft jet engines. For example, various investiga-

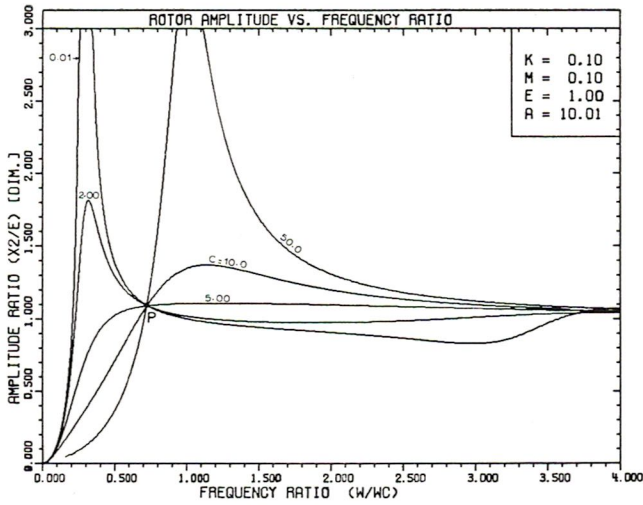


Fig. 12 Rotor amplitude vs. speed for a low mass ratio tuned support system for various values of support damping, $K = M = 0.1$, $A = 10$

$$C_1 = 13.6 \times C_2 = 340 \text{ lb-sec/in.}$$

Note that this value is approximately the same as the value given in Example 2 for the required damping at point P corresponding to A_∞ .

This indicates that each support must have 170 lb-sec/in. damping to achieve the optimum response of about 1.7 times the unbalance level of the rotor.

Next consider a tuned support with a mass and stiffness ratio of 0.10 (see Fig. 12). Corresponding support weight and stiffness are given as follows

$$W_1 = 9.7 \text{ lb/in.}$$

$$K_1 = 25,000 \text{ lb/in.}$$

The required damping is thus found from Fig. 11 to be

$$C = 5.0$$

or

$$C_1 = 5 \times 25 = 125 \text{ lb-sec/in.}$$

Thus only 62.5 lb-sec/in. damping per support is required to obtain an optimum response of 1.1 times the unbalance level of the rotor. This value of 1.1 is in comparison to a maximum response of 10 times the unbalance level for the rigidly mounted rotor-bearing system.

Optimization of Support Damping for Off-Tuned Conditions

In general it is not possible or necessarily desirable to have a tuned support system. The support to rotor mass ratio is usually dictated by design considerations and can be varied only within certain ranges. Fig. 11 shows that for best reduction of rotor amplitude, the support mass should remain as light as possible. However, it will be shown that even with high mass ratio support systems the rotor amplitudes can be attenuated by a factor of 5 by proper selection of the stiffness and damping coefficients.

To evaluate the optimum damping for off-tuned conditions the computer program was run for various support mass and stiffness ratios and each of these for various damping coefficients. For example, Fig. 13 represents the amplitudes at the rotor first and second critical speed for various mass ratios with a dimensionless stiffness ratio of $K = 1$ as the mass ratio and damping are varied. The solid lines represent the amplitude at the second critical speed and dotted lines represent the amplitude at the first critical speed. With moderate support damping ratios it is ob-

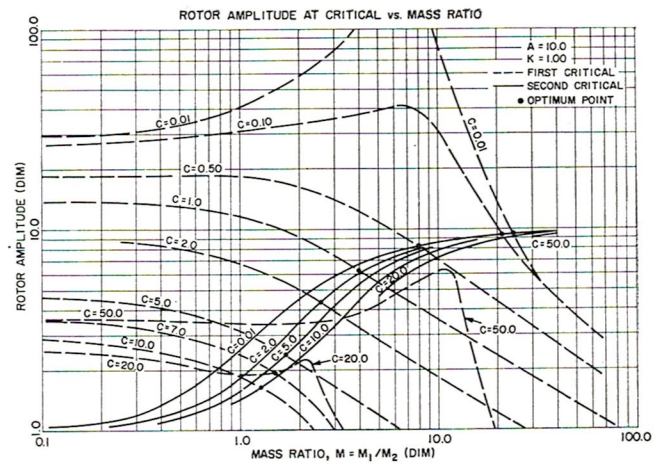


Fig. 13 Rotor amplitude at critical speeds vs. mass ratio for various damping ratios, $A = 10.0$, $K = 1.0$

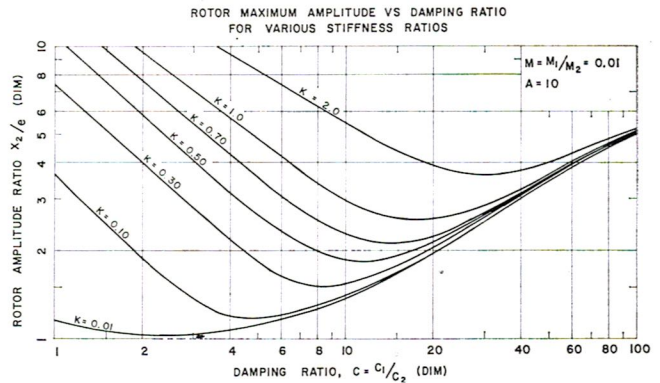


Fig. 14 Rotor maximum amplitude vs. damping ratio for various values of stiffness ratios for a low mass ratio support, $M = 0.01$, $A = 10$

served that as the mass ratio increases the amplitude at the first critical reduces while the amplitude at the second critical increases. The optimum damping was selected as the intersection of the amplitudes at the first and second critical for a particular value of damping. For example, the lowest optimum amplitude point on the plot is given by a damping ratio of 10 and produces an amplitude ratio of about 1.5. Several plots similar to Fig. 13 were produced and the results were then cross-plotted to obtain plots of amplitude versus damping ratio.

Fig. 14 represents the maximum rotor amplitude versus support damping ratio for various values of dimensionless support stiffness for a rotor bearing system with a low support mass ratio of 0.01. Fig. 14 shows that for this particular case, the lowest amplitude is achieved by a low support stiffness ratio of $K = 0.01$ which is of the same order as the mass ratio. With this low support stiffness, there is a wide range of support damping (i.e. $C = 1 \rightarrow 6$) that can be used to achieve the low level of rotor response.

Thus, under proper design conditions the support damping may be allowed to vary by a considerable amount without impairing the rotor performance. As the support stiffness ratio increases, the maximum rotor amplitude response also increases and the required support damping must be larger. For example, if the support stiffness ratio increases from 0.01 to 2.0, the optimum damping required increases by a factor of 15 from approximately 2 to 30.

Note also that for high stiffness ratio support systems, the permissible range of the support damping coefficient is very narrow, and that either a reduction or an increase of damping beyond the optimum value will result in a rapid gain in rotor response.

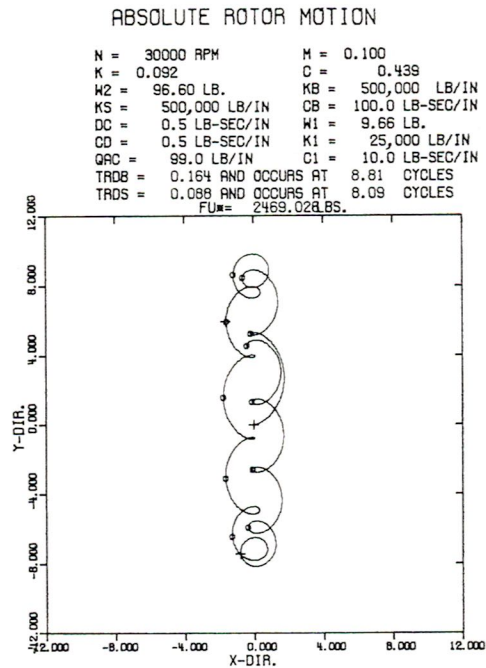


Fig. 18 Dimensionless transient rotor motion with underdamped flexible supports for 12 cycles, $K = M = 0.10$, $C = 0.44$

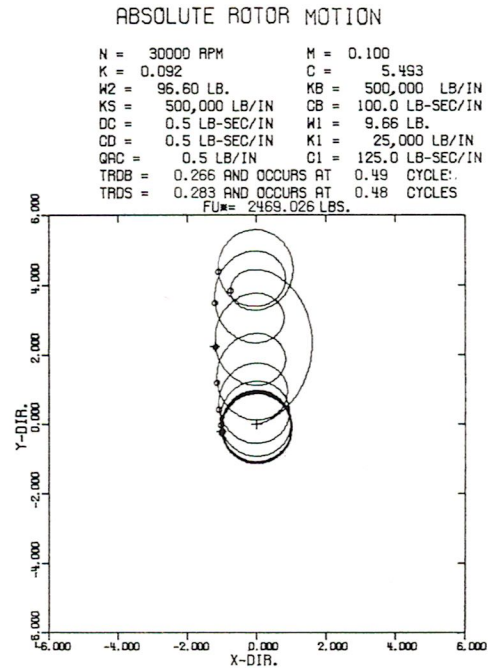


Fig. 19 Dimensionless rotor motion with optimum steady-state damping showing the steady-state orbit after 7 cycles of running speed, $K = M = 0.1$, $C = 5.5$

tors have observed that such a situation occurs with the hydrodynamic squeeze film bearing when operating at excessive eccentricity ratios [29].

Fig. 18 represents the transient orbit for the same rotor system except that the support damping has been reduced by a factor of 100 from $C_1 = 1000$ lb-sec/in. to 10 lb-sec/in. In this case, the maximum force transmitted through the support is less than 9 percent of the rotating unbalance force and the bearing force transmitted is 16 percent. This orbit is analogous to a suddenly applied unbalance such as a blade loss in an engine. Although the forces transmitted have been greatly reduced with the low stiffness and damping support system, the rotor has developed a large initial transient motion of over 10 times the unbalance eccentricity and this transient motion is not readily damped out.

In Fig. 19, the rotor transient motion is depicted with an optimum damping coefficient of $C = 5.5$ for minimum rotor response as determined from the steady-state analysis. The transient response is rapidly suppressed after seven cycles of shaft motion to produce a small stable synchronous orbit. The transmitted forces to the bearings and support are nearly balanced to achieve approximately a 75 percent attenuation of the unbalance load.

Summary and Conclusions

The equations of motion for a single-mass rotor-bearing system on damped flexible supports have been derived and studied considering both a steady-state and transient-type analysis. Design charts for both tuned and off-tuned support conditions have been presented.

The analysis may be summarized by the following general statements:

- 1 The critical speed response of the single-mass Jeffcott model rotor may be completely eliminated by means of a low mass ratio flexible support with optimum damping. In this case the rotor steady-state amplitude of motion over the entire speed range will only be slightly more than the rotor unbalance eccentricity.
- 2 The support mass ratio should be kept as light as possible to achieve minimum rotor amplitude.
- 3 The rotor amplitude may be considerably attenuated even

for high mass ratio support systems by tuning the support stiffness such that $K = M$ and incorporating optimum damping for the tuned conditions.

4 With a low mass ratio support system, the required value of optimum damping is not critical and can vary by a factor of 10 without appreciably effecting rotor performance. As the mass ratio increases, the required value of optimum damping increases rapidly and the permissible range of variation of support damping diminishes.

5 The off-tuned support ($K \neq M$) can be designed to produce a considerable improvement in system response in comparison to the rotor on rigid supports. If insufficient damping is incorporated in the support then the resulting rotor steady-state amplitude may be larger than the original rotor response for support stiffness values $K > 1$.

6 If there is excessive support damping ($C > 20$) with a low mass ratio support ($M = 0.1$), then the forces transmitted through the support may exceed the unbalance forces (TRD's > 1.0).

7 Although the steady-state analysis shows that the rotor amplitude will be small for an underdamped ($C < 0.50$) low mass ratio support system, the orbital analysis shows that a large initial transient motion can be generated due to the suddenly applied unbalance force and that this motion is not readily attenuated.

8 The optimum damping based on minimization of the rotor steady-state amplitude for both tuned and off-tuned conditions produces a satisfactory transient response from the standpoint of rapid reduction of the initial transient motion, improved system stability and reduction of the forces transmitted.

Acknowledgment

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